Non-wellfounded trees in Homotopy Type Theory*

Benedikt Ahrens[1], Paolo Capriotti[2] and Régis Spadotti[1]

[1] Institut de Recherche en Informatique de Toulouse, Université Paul Sabatier
{ahrens,spadotti}@irit.fr
[2] Functional Programming Laboratory, School of Computer Science, University of Nottingham
pvc@cs.nott.ac.uk

Coinductive data types are used in functional programming to represent infinite data structures. Examples include the ubiquitous data type of streams over a given base type, but also more sophisticated types.

From a categorical perspective, coinductive types are characterized by a *universal property*, which specifies the object with that property *uniquely* in a suitable sense. More precisely, a coinductive type is specified as the *terminal coalgebra* of a suitable endofunctor. In this category-theoretic viewpoint, coinductive types are dual to *inductive* types, which are defined as initial algebras.

Inductive, resp. coinductive, types are usually considered in the principled form of the family of W-types, resp. M-types, parametrized by a type A and a dependent type family B over A, that is, a family of types $(B(a))_{a:A}$. Intuitively, the elements of the coinductive type $\mathsf{M}(A,B)$ are trees with nodes labeled by elements of A such that a node labeled by $a : A$ has $B(a)$-many subtrees, given by a map $B(a) \to \mathsf{M}(A,B)$; see Figure 1 for an example. The *inductive* type $\mathsf{W}(A,B)$ contains only trees where any path within that tree eventually leads to a *leaf*, that is, to a node $a : A$ such that $B(a)$ is empty.

$$a, b, c : A \qquad B(a) = 0 \qquad B(b) = 2 \qquad B(c) = 3$$

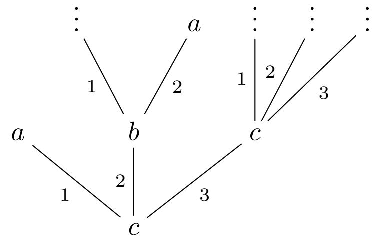

Figure 1: Example of a tree (adapted from [7])

The present work takes place in intensional Martin-Löf type theory extended by Voevodsky's *Univalence Axiom*. We show that, in this type theory, *coinductive* types in the form of M-types can be *derived* from *inductive* types. (More precisely, only one specific W-type is needed: the type of natural numbers, which is readily specified as a W-type [4].) Indeed, given a signature (A,B) specifying a shape of trees as described above, we construct the M-type associated to that signature and prove its universal property. The construction can be seen as a higher-categorical analogue of the classical construction of the terminal coalgebra of some endofunctor as the limit of a chain.

*The work of Benedikt Ahrens was partially supported by the CIMI (Centre International de Mathmatiques et d'Informatique) Excellence program ANR-11-LABX-0040-CIMI within the program ANR-11-IDEX-0002-02 during a postdoctoral fellowship.

The result presented in this work is not surprising: indeed, the constructibility of coinductive types from inductive types has been shown in *extensional* type theory (that is, type theory with identity reflection) [7, 1], as well as in type theory satisfying Axiom K [3]. It was conjectured to work in homotopy type theory, that is, the type theory described in [6], during a discussion on the HoTT mailing list [5].

We have formalized our results in the proof assistant Agda.

The theorem we prove here is actually more general than described above: instead of plain M-types as described above, we construct *indexed* M-types, which can be considered as a form of "simply-typed" trees, typed over a type of indices I. Plain M-types then correspond to the mono-typed indexed M-types, that is, to those for which $I = 1$.

The details of this work are described in an article [2]. The source code and HTML documentation of the Agda formalization can be downloaded from `https://hott.github.io/M-types/`.

References

[1] Michael Abbott, Thorsten Altenkirch, and Neil Ghani. Containers: Constructing strictly positive types. *Theor. Comput. Sci.*, 342(1):3–27, 2005. doi:10.1016/j.tcs.2005.06.002.

[2] Benedikt Ahrens, Paolo Capriotti, and Régis Spadotti. Non-wellfounded trees in Homotopy Type Theory. *To be published in Proceedings of Typed Lambda Calculi and Applications*, 2015. arXiv:1504.02949.

[3] Thorsten Altenkirch, Neil Ghani, Peter Hancock, Conor McBride, and Peter Morris. Indexed Containers. available at `http://www.cs.nott.ac.uk/~txa/publ/jcont.pdf`.

[4] Steve Awodey, Nicola Gambino, and Kristina Sojakova. Inductive types in homotopy type theory. In *Proceedings of Symposium on Logic In Computer Science*, pages 95–104, 2012. doi:10.1109/LICS.2012.21.

[5] HoTT mailing list. Discussion on coinductive types on HoTT mailing list, `https://groups.google.com/d/msg/homotopytypetheory/tYRTcI2Opyo/PIrI6t5me-oJ`.

[6] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics.* `http://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

[7] Benno van den Berg and Federico De Marchi. Non-well-founded trees in categories. *Ann. Pure Appl. Logic*, 146(1):40–59, 2007. arXiv:math/0409158.