Univalent Foundations and the Equivalence Principle

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Outline

1. The equivalence principle
2. Invariance in Univalent Foundations

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Notion of equivalence depends on the objects under consideration:

- **isomorphic** sets, groups, rings,. . .
- **equivalent** categories
- **biequivalent** bicategories
- . . .

Non-examples: statements violating equivalence principle

We can easily **violate** this principle:

Exercise

Find a statement that is not invariant under the equivalence of categories

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A solution

"The category $\mathcal{C}$ has exactly one object."

Maybe this statement is simply silly!

A language for invariant properties

M. Makkai, *Towards a Categorical Foundation of Mathematics*:

> *The basic character of the Principle of Isomorphism is that of a* ***constraint on the language*** *of Abstract Mathematics; a welcome one, since it provides for the separation of sense from nonsense.*

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Goal

to have a **syntactic criterion** for properties and constructions that are invariant under equivalence

How to break the invariance principle for categories…

- Recall: the statement

  The category $\mathcal{C}$ has exactly one object.

  is not invariant under equivalence of categories.
- In general, referring to **equality of objects breaks invariance**, but…
- even the **definition** of category refers to equality of objects:

Problem

"If $\mathsf{source}(g)$ is equal to $\mathsf{target}(f)$, then $g \circ f$ exists."

Can we give a definition of category without using equality of objects?

. . . and how to fix it.

Solution

Use a logic/language of **dependent types**, in which $s(g) = t(f)$ is encoded by what type of thing f and g are.

A category consists of

- A collection O of objects
- For each $x, y : O$, a collection $A(x, y)$ of arrows
- For each $x, y, z : O$ and each $f : A(x, y)$ and $g : A(y, z)$, a composite $g \circ f : A(x, z)$
- For each $x : O$, an identity $1_x : A(x, x)$
- . . .

Gives rise to **dependently typed language** by adding logical connectors.

Invariance for statements

Theorem (Freyd 76, Blanc 78)

A property of categories is invariant under equivalence iff it can be expressed in this dependently typed language, using equality for arrows but not for objects.

Now:

- Extend this result to **constructions on categories**...
- ...in a way a computer can understand.

Outline

Univalent Foundations

- A language of **dependent types**, a.k.a. a **type theory**
- With an interpretation in ($\infty-$)groupoids

Type theory	Interpretation
type A	groupoid A
term a of type A	object a of groupoid A
function $A \to B$	∞-functor $A \to B$

- Universe of **sets** given by **discrete groupoids**
- Logic and constructions are treated uniformly in UF

The Univalence Axiom

Definition

A map $f : A \to B$ of types is an **equivalence** if there is $g : B \to A$ such that

- for any $a : A, g(f(a)) \simeq a$
- for any $b : B, f(g(b)) \simeq b$

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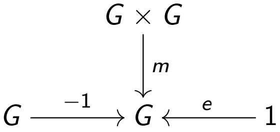
Univalence Axiom: equivalence principle for types

An equivalence of types lifts to an equivalence of constructions on those types.

A **group** in Univalent Foundations is a group object, i.e.



such that

- G is a **discrete** type (groupoid), i.e. a "set"
- obvious group axioms are satisfied

Lifting the equivalence principle to algebraic structures

A group isomorphism $G \to G'$ is

- a bijective function on the underlying types $G \to G'$
- compatible with the group structures on G and G'.

EP on types lifts to EP on groups:

Structure Identity Principle (Aczel, Coquand, Danielsson)

- An iso of groups lifts to an equivalence of constructions on groups in UF.
- In particular: any statement about groups in UF is invariant under group iso.
- and similar for other algebraic structures

An equivalence principle for categories

Express Structure Identity Principle differently:

SIP categorically:

In the categories of sets, groups, rings,. . . , any construction expressible in UF is invariant under **isomorphism**.

Going higher:

Theorem (A., Kapulkin, Shulman)

*In the bicategory of **saturated** categories, any construction in Univalent Foundations is invariant under **equivalence**.*

What is this **saturation condition**?

Saturation: internalizing categories in groupoids

For any category C, its **saturation** is a category $\hat{C} = A \rightrightarrows O$ internal to groupoids:

- O is the groupoid of objects and isomorphisms of C
- A is the groupoid of arrows and isomorphisms (commutative squares) of C

Fact

$F : C \to D$ is an equivalence iff $\hat{F} : \hat{C} \to \hat{D}$ is an equivalence on underlying groupoids (i.e. $O_{\hat{C}} \to O_{\hat{D}}$ and $A_{\hat{C}} \to A_{\hat{D}}$ are equivalences).

Categories in groupoids

Theorem (M. Shulman)

An internal category $A \rightrightarrows O$ *in groupoids is of the form* $\hat{C}$ *iff*

- *the map* $A \to O \times O$ *is a discrete fibration, and*
- *the canonical map*

$$\mathsf{hom}_O(x, y) \to \mathsf{iso}(x, y)$$

is an isomorphism for all $x, y \in \mathsf{ob}(O)$.

- $\mathsf{hom}_O(x, y)$ = morphisms in the groupoid O
- $\mathsf{iso}(x, y)$ = objects of the groupoid A that are isomorphisms from x to y for the category structure on $A \rightrightarrows O$.

Such a category in groupoids is called univalent (or “saturated” or “Rezk-complete”).

The equivalence principle for saturated categories

Theorem

In Univalent Foundations, any construction and property of a ***saturated*** *category* $A \rightrightarrows O$ *is invariant under equivalence.*

Examples of saturated categories

- Set
- Grp, Rng,… (categories of algebraic structures)
- $[\mathcal{C}, \mathcal{D}]$, if $\mathcal{D}$ is
- any full subcategory of a saturated category

The Rezk completion

Not every internal category is of the form $\hat{C}$, but

Theorem (A., Kapulkin, Shulman)

*Any internal category $D = A \rightrightarrows O$ with discrete fibers is **weakly equivalent** to a saturated one—its **Rezk completion**.*

- "Weakly equivalent" means "there is a fully faithful and essentially surjective functor"
- Proof was verified by computer proof assistant based on Univalent Foundations.
- Special case: for $D = A \rightrightarrows O$ an equivalence relation, the Rezk completion of D is the **quotient** of D.

References

- A., Kapulkin, Shulman: *Univalent categories and the Rezk completion*
- Blanc: *Équivalence naturelle et formules logiques en théorie des catégories*
- Coquand, Danielsson: *Isomorphism is equality*
- Freyd: *Properties invariant within equivalence types of categories*
- Makkai: *Towards a Categorical Foundation of Mathematics*
- Rezk: *A model for the homotopy theory of homotopy theory*
- Voevodsky: *Univalent Foundations project*