Categorical structures in type theory, in (univalent) type theory

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Goal

Compare various categorical structures arising in study of type theory:

- categories with families
- (split) type categories
- categories with display maps

Remarks:

- In general, we will compare structures on a fixed base category.
  E.g., we consider $\mathsf{FamStr}(\mathcal{C})$ the family structures on fixed category $\mathcal{C}$, etc.

What are those categorical structures?

- gadgets are sometimes used to interpret syntax
- additional structure on those gadgets is often considered (Pi, Id,...)

- This talk is not about syntax nor about interpretation of syntax.
- In this talk additional structure is not considered.

For all this, see Lumsdaine's talks.

How to compare?

- in set theory and traditional ML type theory, comparison on the level of categories—i.e., up to **isomorphism**
- in **univalent** type theory, can meaningfully compare on level of objects—i.e., up to **identity**

Refined goal: compare in (univalent) type theory and formalize in `UniMath`.

The type theory we work in—UniMath

- Martin-Löf type theory
- identity types
- sigma types
- dependent function types
- natural numbers, booleans
- disjoint sum types (could be replaced by sigma)
- propositional truncation (via impredicative encoding)
- $\mathcal{U} : \mathcal{U}$
- univalence axiom for $\mathcal{U}$

For the purpose of formalization, the fragment of the Coq proof assistant that constitutes the `UniMath` language

Outline

1. Review of categories in univalent type theory

2. Categorical structures in type theory
   - Families and split type structures
   - Families and Rezk completion

Outline

Categories in univalent mathematics

A category is

- a type $O : \mathcal{U}$ of objects
- a dependent type $A : O \times O \to \mathcal{S}et$ of arrows
- $\mathsf{id} : \prod_{(a:O)} A(a,a)$
- $(\circ) : \prod_{(a,b,c:O)} A(a,b) \times A(b,c) \to A(a,c)$
- axioms postulating identities of arrows

Categories in univalent mathematics

A **univalent** category is

- a type $O : \mathcal{U}$ of objects
- a dependent type $A : O \times O \to \mathcal{S}et$ of arrows
- $\mathsf{id} : \prod_{(a:O)} A(a,a)$
- $(\circ) : \prod_{(a,b,c:O)} A(a,b) \times A(b,c) \to A(a,c)$
- axioms postulating identities of arrows
- such that the natural map

$$\mathsf{idtoiso} : \prod_{a,b:O} (a = b) \to \mathsf{iso}(a,b)$$

is an equivalence for any a, b

Sameness for univalent categories

$$\mathcal{C} = \mathcal{D}$$
$$\mathcal{C} \cong \mathcal{D}$$
$$\mathcal{C} \simeq \mathcal{D}$$

Theorem

For ***univalent*** *categories* $\mathcal{C}$ *and* $\mathcal{D}$*, these are equivalent as types.*

Consequence

Every property of univalent categories definable in UF is invariant under equivalence.

Examples of univalent categories

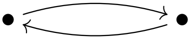
- $\mathcal{S}et$ (discrete types)
- Groups, rings, ... (Structure Identity Principle)
- Functor category $[\mathcal{C}, \mathcal{D}]$, if $\mathcal{D}$ is univalent
- Full subcategories of univalent categories

More examples of univalent categories

- A preorder is univalent iff it is antisymmetric
- If X is of h-level 3, i.e., 1-truncated, then there is a univalent category with X as objects and $\hom(x, y) := (x = y)$
- If $\mathcal{C}$ is univalent, then the category of cones of shape $F : \mathcal{J} \to \mathcal{C}$ is
  - $\rightsquigarrow$ limits (limiting cones) in a univalent category are unique **up to identity**

Non-univalent categories

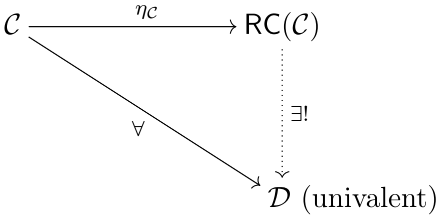
- Any "chaotic" category $\mathcal{C}$ with $\mathcal{C}(x, y) := 1$, for $\mathcal{C}_0$ not a prop



- Any chaotic category $\mathcal{C}$ with an object $c : \mathcal{C}_0$ is **equivalent** to the terminal category $\mathbf{1}$
  - $\rightsquigarrow$ a category can be equivalent to a univalent one without being univalent itself

Rezk completion

To any category $\mathcal{C}$, associate a univalent one, its "Rezk completion",



Intuitively, obtain $\mathsf{RC}(\mathcal{C})_0$ by adding to $\mathcal{C}_0$ as many identities as needed

Construction of the Rezk completion

- $\mathsf{RC}(\mathcal{C})$ is the full image subcategory of the Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{S}et]$
- $\eta_{\mathcal{C}} : \mathcal{C} \to \mathsf{RC}(\mathcal{C})$ is fully faithful and essentially surjective
- precomposition with a ff. and es. functor is ff. and es.
- a ff. and es. functor is an equivalence if source category is univalent
- the object map of an equivalence of univalent categories is an equivalence of types

About the Rezk completion

- Name inspired by
  Charles Rezk, *A model for the homotopy theory of homotopy theory*
  categories $\sim$ truncated Segal spaces
  univalent categories $\sim$ complete truncated Segal spaces
  in Voevodsky's simplicial set model

- More details in Ahrens, Kapulkin, Shulman, *Univalent categories and the Rezk completion*
  `http://arxiv.org/abs/1303.0584`

Outline

Rezk completion and categorical structures of type theory

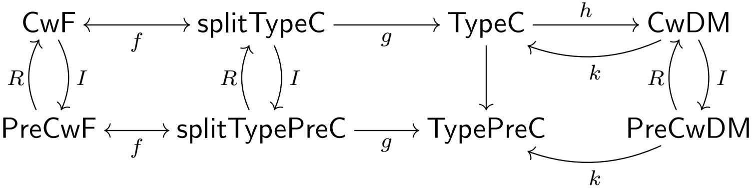
Questions

1. Does a given structure on $\mathcal{C}$, e.g., a CwF-structure, induce a structure on $\mathsf{RC}(\mathcal{C})$?
2. When can we construct a map

$$s_1(\mathcal{C}) \to s_2(\mathcal{C})$$

turning one kind of structure into another kind?
[Answer: may depend on $\mathcal{C}$ being univalent!]

Goal



- These are maps of types
- f, g, h, k do not change the base category
- g is injective (forgets splitness)
- f is an equivalence

Categories with Families

A **families structure** on a category $\mathcal{C}$ is

- presheaves $\mathsf{Ty}, \mathsf{Tm} : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$;
- a natural transformation $\mathsf{p} : \mathsf{Tm} \to \mathsf{Ty}$;
- for any $\Gamma : \mathcal{C}$ and $A : \mathsf{Ty}(\Gamma)$, an object $\Gamma.A$ and a map $\pi_A : \mathcal{C}(\Gamma.A, \Gamma)$
- for any $\Gamma : \mathcal{C}$ and $A : \mathsf{Ty}(\Gamma)$, a map Q_A making a pullback square

$$\begin{array}{ccc}
\mathsf{Yo}(\Gamma.A) & \xrightarrow{\mathsf{Q}_A} & \mathsf{Tm} \\
{\scriptstyle \mathsf{Yo}(\pi_A)}\downarrow & & \downarrow{\scriptstyle \mathsf{p}} \\
\mathsf{Yo}(\Gamma) & \xrightarrow[\mathsf{v}(A)]{} & \mathsf{Ty}
\end{array}$$

Type of such structures is called $\mathsf{FamStr}(\mathcal{C})$

Split type categories

A **split type structure** on a category $\mathcal{C}$ is

- a presheaf $\mathsf{Ty} : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$
- for any $\Gamma : \mathcal{C}_0$ and $A : \mathsf{Ty}(\Gamma)$, an object $\Gamma.A$ and $\pi_A : \mathcal{C}(\Gamma.A, \Gamma)$
- for any $\Gamma, \Gamma' : \mathcal{C}_0$ and $f : \mathcal{C}(\Gamma', \Gamma)$ and $A : \mathsf{Ty}(\Gamma)$, a morphism $\mathsf{q}_{(f,A)} : \mathcal{C}(\Gamma.A, \Gamma)$ such that

$$\begin{array}{ccc}
\Gamma'.A[f] & \xrightarrow{\mathsf{q}_{(f,A)}} & \Gamma.A \\
{\scriptstyle \pi_{A[f]}}\downarrow & & \downarrow{\scriptstyle \pi_A} \\
\Gamma' & \xrightarrow[f]{} & \Gamma
\end{array}$$

  is a pullback
- axioms about functoriality of q

Split type categories

"Functoriality" of q:

- $\mathsf{q}_{(1,A)} = \mathsf{idtoiso}(\mathsf{ap}_{x \mapsto \Gamma.x}(p))$
- $\mathsf{q}_{(f' \cdot f, A)} = \mathsf{idtoiso}(\mathsf{ap}_{x \mapsto \Gamma''.x}(p')) \cdot \mathsf{q}_{(f', A[f])} \cdot \mathsf{q}_{(f,A)}$

where $p : A[1] = A$ and $p' : A[f' \cdot f] = A[f][f']$

- Type of such structures is called $\mathsf{sTypeStr}(\mathcal{C})$
- Called CwA in Lumsdaine's talk

Constructions

Construction 1

An equivalence

$$\mathsf{sTypeStr}(\mathcal{C}) \simeq \mathsf{FamStr}(\mathcal{C})$$

Here, $\simeq$ is equivalence of both types and categories

Construction 2

A function

$$\mathsf{FamStr}(\mathcal{C}) \to \mathsf{FamStr}(\mathsf{RC}(\mathcal{C}))$$

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Goal

To construct an equivalence of types

$$\mathsf{FamStr}(\mathcal{C}) \simeq \mathsf{sTypeStr}(\mathcal{C})$$

Sketch of the equivalence

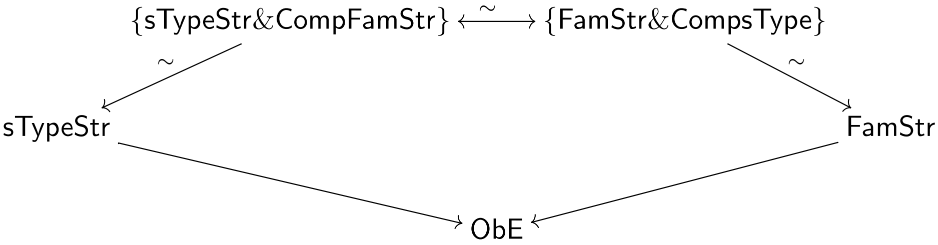
$$\begin{array}{ccc}
\{\mathsf{sTypeStr}\&\mathsf{CompFamStr}\} & \stackrel{\sim}{\longleftrightarrow} & \{\mathsf{FamStr}\&\mathsf{CompsTypeStr}\} \\
\sim\downarrow & & \downarrow\sim \\
\mathsf{sTypeStr} & & \mathsf{FamStr}
\end{array}$$

- vertical maps are first projections—we show that fibers are contractible
- horizontal map is rearranging of components
- need univalence to show that left projection is equivalence
- need function ext. for right projection being an equivalence

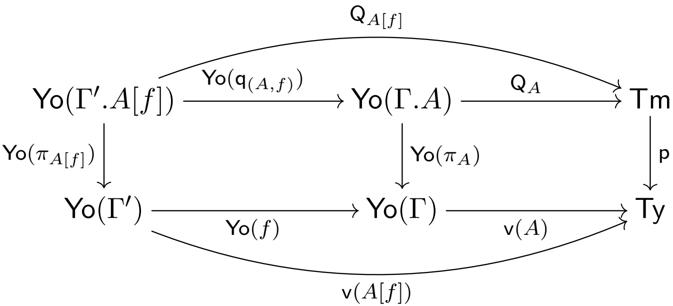
Some details

Fiberwise over common elements of $\mathsf{FamStr}(\mathcal{C})$ and $\mathsf{sTypeStr}(\mathcal{C})$:
An **object extension structure** on a category $\mathcal{C}$ is given by

- a functor $\mathsf{Ty} : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$
- for any $\Gamma : \mathcal{C}_0$ and $A : \mathsf{Ty}(\Gamma)$, an object $\Gamma.A$ a morphism $\pi_A : \mathcal{C}(\Gamma.A, \Gamma)$



Compatibility between families and split type structure



Compatibility is commutativity of upper triangle

$$\mathsf{Q}_{A[f]} = \mathsf{Yo}(\mathsf{q}(A, f)) \cdot \mathsf{Q}_A$$

Categorical equivalence

Also construct a categorical equivalence

$$\mathsf{FamStr}(\mathcal{C}) \simeq \mathsf{sTypeStr}(\mathcal{C})$$

- does not require univalence axiom
- makes use of notion of **displayed categories** to work fiberwise over a fixed category of object extension structures

Type-theoretic and categorical equivalence

Lemma

If $F : \mathcal{C} \simeq \mathcal{D}$ is an equivalence and $\mathcal{C}$ and $\mathcal{D}$ are univalent, then $F_0 : \mathcal{C}_0 \simeq \mathcal{D}_0$.

- recover equivalence of types $\mathsf{FamStr}(\mathcal{C}) \simeq \mathsf{sTypeStr}(\mathcal{C})$ by showing that $\mathsf{FamStr}(\mathcal{C})$ and $\mathsf{sTypeStr}(\mathcal{C})$ are univalent categories
- proving that categories $\mathsf{FamStr}(\mathcal{C})$ and $\mathsf{sTypeStr}(\mathcal{C})$ are univalent requires univalence axiom

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Goal

To construct a function

$$\mathsf{FamStr}(\mathcal{C}) \to \mathsf{FamStr}(\mathsf{RC}(\mathcal{C}))$$

- can be done directly via universal property of $\mathsf{RC}(\mathcal{C})$
- is cumbersome

Generalize ...

Universe relative to a functor

A **universe structure** relative to a functor $J : \mathcal{C} \to \mathcal{D}$ is given by

- objects U and $\tilde{\mathsf{U}}$ in $\mathcal{D}$ and a morphism $\mathsf{p} : \tilde{\mathsf{U}} \to \mathsf{U}$;
- for any object X in $\mathcal{C}$ and any morphism $f : J(X) \to \mathsf{U}$,
  - an object X_f in $\mathcal{C}$
  - a morphism $\mathsf{p}_f : X_f \to X$
  - a morphism $\mathsf{q}_f : J(X_f) \to \tilde{\mathsf{U}}$ such that

$$\begin{array}{ccc} J(X_f) & \xrightarrow{\mathsf{q}_f} & \tilde{\mathsf{U}} \\ {\scriptstyle J(\mathsf{p}_f)}\downarrow & & \downarrow{\scriptstyle \mathsf{p}} \\ J(X) & \xrightarrow[f]{} & \mathsf{U} \end{array}$$

  is a pullback square.

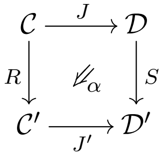
Relative universe on Yoneda, identity functors

- Denote $\mathsf{Yo} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{S}et]$ the Yoneda functor. Then

$$\mathsf{RelUniv}(\mathsf{Yo}) \simeq \mathsf{FamStr}(\mathcal{C})$$

- $J = \mathrm{Id}(\mathcal{C}) \quad \rightsquigarrow \quad$ universe category (Voevodsky, *A C-system defined by a universe category*)

Transfer of relative universe structures

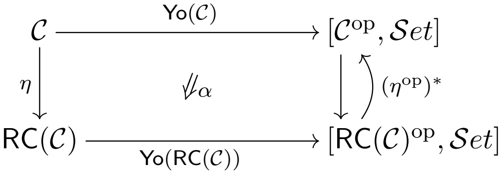


- $\mathcal{C}', \mathcal{D}'$ are univalent
- R is essentially surjective,
- S is fully faithful and preserves pullback squares
- J' is fully faithful
- α is iso

Construction

$$\mathsf{RelUniv}(J) \to \mathsf{RelUniv}(J')$$

Rezk completion and families



Construction

$$\mathsf{FamStr}(\mathcal{C}) \simeq \mathsf{RelUniv}(\mathsf{Yo}(\mathcal{C})) \to \mathsf{RelUniv}(\mathsf{Yo}(\mathsf{RC}(\mathcal{C}))) \simeq \mathsf{FamStr}(\mathsf{RC}(\mathcal{C}))$$

Some remarks

Formalization:

- ca. 4000loc, some of it general purpose stuff
- not public yet

Future work:

- universal property of completion of a category with families
- connection to C-systems (formalized by Voevodsky)
- consider additional structure (Pi, Sigma, Id,...)

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- consider additional structure (Pi, Sigma, Id,...)

Thanks for your attention.

References

- First occurrence of univalent categories: Hofmann & Streicher, *The groupoid interpretation of type theory*
- Structure Identity Principle: Coquand & Danielsson, *Isomorphism is equality*
- (Complete) segal spaces: Rezk, *A model for the homotopy theory of homotopy theory*
- Categories with families: Fiore, *Discrete Generalised Polynomial Functors*, Talk at ICALP 2012 (Appendix) `http://www.cl.cam.ac.uk/~mpf23/talks/ICALP2012.pdf`
- (Split) type categories: van den Berg & Garner, *Topological and simplicial models*
  also Pitts, *Categorical Logic*
- Universe categories: Voevodsky, *A C-system defined by a universe category*