Univalent categories and the Rezk completion

Benedikt Ahrens

joint work with Krzysztof Kapulkin and Michael Shulman

Institut de Recherche en Informatique de Toulouse
Université Paul Sabatier

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3 kinds of sameness for categories

Equality	$\mathcal{C} = \mathcal{D}$
Isomorphism	$\mathcal{C} \cong \mathcal{D}$
Equivalence	$\mathcal{C} \simeq \mathcal{D}$

- most properties of categories invariant under **equivalence**
- we can only substitute **equals for equals**
- in set-theoretic foundations these notions are worlds apart

In this talk:

Define categories in the **Univalent Foundations** for which all three coincide

Outline

Univalent Foundations

What are the Univalent Foundations?

- Intensional Martin-Löf Type Theory

$\rightsquigarrow$ *Types as Spaces* interpretation, i.e. Homotopy Type Theory

$+$ **Univalence Axiom**

Martin-Löf Type Theory and its Homotopy Interpretation

Sigma type	$\sum_{x:A} B(x)$	total space of a fibration
Product type	$\prod_{x:A} B(x)$	space of sections of a fibration
Identity type	$\mathrm{Id}_A(a, b)$	space of **paths** $p : a \rightsquigarrow b$

Level of a type

Definition (Propositions & Sets)

A type A is a **proposition** if any two $a, b : A$ are equal, that is,

$$\mathrm{isProp}(A) := \prod_{x\ y:A} \mathrm{Id}(x, y)$$

A type A is a **set** if for any $x, y : A$, the type $\mathrm{Id}(x, y)$ is a proposition

$$\mathrm{isSet}(A) := \prod_{x\ y:A} \mathrm{isProp}(\mathrm{Id}(x, y))$$

- Propositions are "proof–irrelevant" types.
- Points of a set are equal in a unique way, if they are.

Univalence : equivalent types are equal

Universes in MLTT

- Types in MLTT are stratified in **Universes** $\mathcal{U}_n$
- can consider $\mathrm{Id}_{\mathcal{U}}(A, B)$ (polymorphic in universe level n)
- **Univalence** allows to construct identities between A and B

Univalence: short description

- Define type $\mathrm{Equiv}(A, B)$ of **Equivalences from A to B**
- Univalence Axiom identifies $\mathrm{Equiv}(A, B)$ with $\mathrm{Id}(A, B)$
- Can construct $f : \mathrm{Equiv}(A, B)$ for suitable A, B

Equivalence of types

Definition (Equivalence of types)

A function $f : A \to B$ is an **equivalence of types** if there are

- $$g : B \to A$$
- $$\eta : \prod_{a:A} \mathrm{Id}\Big(g\big(f(a)\big), a\Big) \qquad \epsilon : \prod_{b:B} \mathrm{Id}\Big(f\big(g(b)\big), b\Big)$$

together with a coherence condition $\tau : \prod_{x:A} \mathrm{Id}\Big(f(\eta x), \epsilon(fx)\Big)$

Lemma

"f is an equivalence" is a proposition, written isEquiv(f)

The Univalence Axiom

Definition (From paths to equivalences)

$$\mathrm{Equiv}(A, B) := \sum_{f : A \to B} \mathrm{isEquiv}(f)$$

$$\begin{aligned} \mathrm{id\_to\_equiv}_{A,B} : \mathrm{Id}(A, B) &\to \mathrm{Equiv}(A, B) \\ \mathrm{refl}(A) &\mapsto (a \mapsto a) \end{aligned}$$

Univalence Axiom

$$\mathrm{univalence} : \prod_{A\ B : \mathcal{U}} \mathrm{isEquiv}(\mathrm{id\_to\_equiv}_{A,B})$$

Consequences of Univalence

- Equivalent types are equal

$$\text{equiv\_to\_id}_{A,B} : \text{Equiv}(A, B) \to \text{Id}(A, B)$$

- Propositional extensionality

$$(P \leftrightarrow Q) \to \text{Id}(P, Q)$$

- Function extensionality (also for dependent functions)

$$\prod_{x:A} \text{Id}_B(fx, gx) \to \text{Id}_{A \to B}(f, g)$$

- Quotient types exist (cf. later)

Outline

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1. Introduction to Univalent Foundations
2. Category Theory in Univalent Foundations

Notation

Write $p : x \rightsquigarrow y$ for $p : \mathrm{Id}_A(x, y)$

Categories in Univalent Foundations — Take I

A naïve definition of categories

A **category** $\mathcal{C}$ is given by

- a type $\mathcal{C}_0$ of **objects**
- for any $a, b : \mathcal{C}_0$, a type $\mathcal{C}(a, b)$ of **morphisms**
- operations: identity & composition

$$\mathrm{id}_a : A(a,a)$$
$$(\circ)_{a,b,c} : A(b,c) \to A(a,b) \to A(a,c)$$

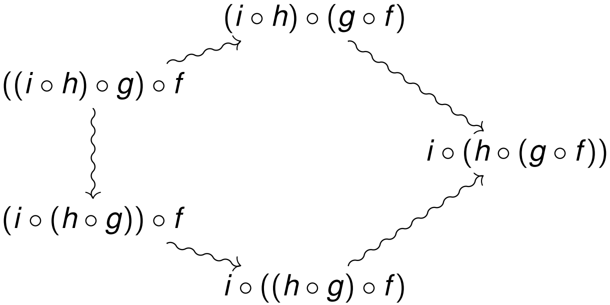
- axioms: unitality & associativity

$$\mathrm{id} \circ f \rightsquigarrow f \qquad f \circ \mathrm{id} \rightsquigarrow f \qquad (h \circ g) \circ f \rightsquigarrow h \circ (g \circ f)$$

Problem: Would require higher coherence data...

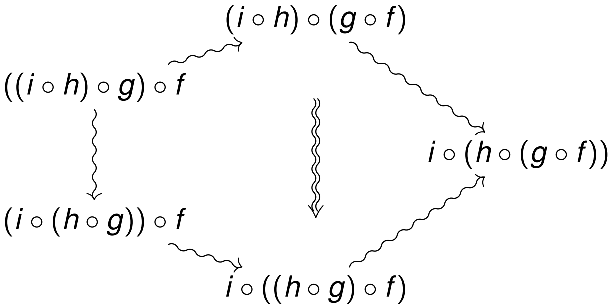
Coherence for associativity

Two ways to associate a composition of **four** morphisms from left to right:



Coherence for associativity

Two ways to associate a composition of **four** morphisms from left to right:



Would need to ask for higher coherence $\rightsquigarrow\!\!\rightsquigarrow$, $\rightsquigarrow\!\!\rightsquigarrow\!\!\rightsquigarrow$ etc

Categories in Univalent Foundations — Take II

A less naïve definition of categories

A **category** $\mathcal{C}$ is given by

- a type $\mathcal{C}_0$ of objects
- for any $a, b : \mathcal{C}_0$, a **set** $\mathcal{C}(a, b)$ of morphisms
- operations: identity & composition
- axioms: unitality & associativity

For this definition of category, the pentagon is automatically coherent.

Isomorphism in a category

Definition (Isomorphism in a category)

A morphism $f : \mathcal{C}(a, b)$ is an **isomorphism** if there are

- $$g : \mathcal{C}(b, a)$$
- $$\eta : g \circ f \rightsquigarrow \mathrm{id}_a \qquad \epsilon : f \circ g \rightsquigarrow \mathrm{id}_b$$

Put differently, we define

$$\mathrm{isIso}(f) := \sum_{g:\mathcal{C}(b,a)} \Big((g \circ f \rightsquigarrow \mathrm{id}_a) \times (f \circ g \rightsquigarrow \mathrm{id}_b) \Big)$$

Isomorphism in a category II

Lemma

For any $f : \mathcal{C}(a, b)$*, the type* $\mathsf{isIso}(f)$ *is a proposition. In particular, the inverse is unique if it exists.*

Definition (The type of isomorphisms)

$$\mathsf{Iso}(a, b) := \sum_{f:\mathcal{C}(a,b)} \mathsf{isIso}(f)$$

What about categories as objects?

Definition (Functor)

A **functor** F from $\mathcal{C}$ to $\mathcal{D}$ is given by

- a function $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$
- for any $a, a' : \mathcal{C}_0$, a function $F_{a,a'} : \mathcal{C}(a, a') \to \mathcal{D}(Fa, Fa')$
- preserving identity and composition

Definition (Isomorphism of categories)

A functor F is an **isomorphism of categories** if

- F_0 is an isomorphism of types and
- $F_{a,a'}$ is an isomorphism of types (a bijection) for any a, a',

$$\mathcal{C} \cong \mathcal{D} := \sum_{F:\mathcal{C}\to\mathcal{D}} \mathrm{isIsoOfCats}(F)$$

Natural transformations

Definition (Natural transformation)

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** $\alpha : F \to G$ is given by

- for any $a : \mathcal{C}_0$ a morphism $\alpha_a : \mathcal{D}(Fa, Ga)$ s.t.
- for any $f : \mathcal{C}(a, b)$, $Gf \circ \alpha_a \rightsquigarrow \alpha_b \circ Ff$

The type of natural transformations $F \to G$ is a **set.**

Definition (Functor category $\mathcal{D}^{\mathcal{C}}$)

- objects: functors from $\mathcal{C}$ to $\mathcal{D}$
- morphisms from F to G: natural transformations

Equivalence of categories

Definition (Left Adjoint)

A functor $F : \mathcal{C} \to \mathcal{D}$ is a **left adjoint** if there are

- $G : \mathcal{D} \to \mathcal{C}$
- $\eta : 1_{\mathcal{C}} \to GF$
- $\epsilon : FG \to 1_{\mathcal{D}}$
- + higher coherence data.

Equivalence of categories

Definition (Equivalence of categories)

A left adjoint F is an **equivalence of categories** if η and ϵ are isomorphisms.

Lemma

"F is an equivalence" is a proposition.

Definition

$$\mathcal{C} \simeq \mathcal{D} := \sum_{F:\mathcal{C}\to\mathcal{D}} \text{isEquivOfCats}(F)$$

From paths to isomorphisms

Definition (From paths to isomorphisms, univalent categories)

For objects $a, b : \mathcal{C}_0$ we define

$$\begin{aligned} \mathrm{id\_to\_iso}_{a,b} : (a \rightsquigarrow b) &\to \mathrm{Iso}(a, b) \\ \mathrm{refl}(a) &\mapsto \mathrm{id}_a \end{aligned}$$

We call the category $\mathcal{C}$ **univalent** if, for any objects $a, b : \mathcal{C}_0$,

$$\mathrm{id\_to\_iso}_{a,b} : (a \rightsquigarrow b) \to \mathrm{Iso}(a, b)$$

is an equivalence of types.

- In a univalent category, isomorphic objects are equal.
- *"$\mathcal{C}$ is univalent"* is a proposition, written $\mathrm{isUniv}(\mathcal{C})$.

Examples of univalent categories

- *Set* (follows from the Univalence Axiom)
- categories of algebraic structures (groups, rings,...)
  - made precise by the **Structure Identity Principle** (Coquand, Aczel)
- full subcategories of univalent categories
- functor category $\mathcal{D}^{\mathcal{C}}$, if $\mathcal{D}$ is univalent

1 kind of sameness for univalent categories

Equality	$\mathcal{C} \rightsquigarrow \mathcal{D}$
Isomorphism	$\mathcal{C} \cong \mathcal{D}$
Equivalence	$\mathcal{C} \simeq \mathcal{D}$

Theorem

*For **univalent** categories $\mathcal{C}$ and $\mathcal{D}$, these three are equivalent as types.*

In particular, we can substitute a univalent category with an equivalent one.

Rezk completion

- *"Being univalent"* is a proposition

$\rightsquigarrow$ Inclusion from univalent categories to categories

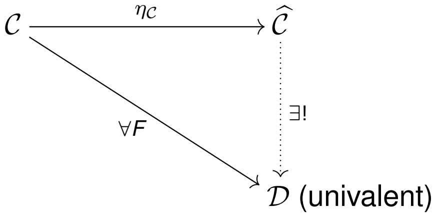
Theorem

The inclusion of univalent categories into categories has a left adjoint (in bicategorical sense),

$$\mathcal{C} \mapsto \widehat{\mathcal{C}}, \qquad \textbf{\textit{Rezk completion}} \textit{ of } \mathcal{C}$$

Rezk completion II

Any functor $F : \mathcal{C} \to \mathcal{D}$ with $\mathcal{D}$ univalent factors uniquely:



The functor $\eta_{\mathcal{C}}$ is the unit of the adjunction; it is

- fully faithful and
- essentially surjective.

Construction of the Rezk completion

- $\widehat{\mathcal{C}} :=$ full image subcat. of $\underline{\mathit{Set}}^{\mathcal{C}^{\mathrm{op}}}$ of **Yoneda embedding**
  - $\widehat{\mathcal{C}}$ is univalent
- let $\eta_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ be the **Yoneda embedding** (into $\widehat{\mathcal{C}}$):
  - fully faithful
  - essentially surjective (by definition)
- precomposition $\_ \circ H : \mathcal{C}^{\mathcal{B}} \to \mathcal{C}^{\mathcal{A}}$ is an equivalence—and hence an isomorphism—of categories if
  - H is essentially surjective
  - $\mathcal{C}$ is univalent
- the object function thus is an isomorphism of types

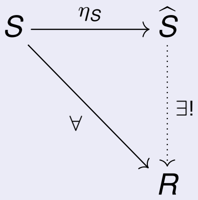
$$\_ \circ H : (\mathcal{C}^{\mathcal{B}})_0 \to (\mathcal{C}^{\mathcal{A}})_0$$

Special case of Rezk completion: Quotienting

Specialise: category $\rightsquigarrow$ groupoid $\rightsquigarrow$ equivalence relation

Theorem

Univalent Foundations admits quotients, i.e. any map $f : S \to R$ *such that* $s \sim s' \implies f(s) = f(s')$ *factors uniquely via* $\widehat{S}$*:*



- More direct construction of set-level quotients by Voevodsky: "type of equivalence classes"

Formalization & future work

Formalization in Coq

- Rezk completion formalized
- approx. 4000 lines of code
- based on Voevodsky's library "*Foundations*"

Work in progress: internal logic of a topos, formalized

- many toposes are univalent

$\rightsquigarrow$ can define a topos as a univalent category with structure

- structure is uniquely determined for univalent categories (not just up to iso)

References

- Univalent Foundations program, *Homotopy Type Theory: Univalent Foundations of Mathematics*, 2013
- M. Hofmann & T. Streicher, *The groupoid interpretation of type theory*, 1996
- C. Rezk, *A model for the homotopy theory of homotopy theory*, 2001
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Thanks for your attention!