Univalent Foundations and the equivalence principle

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Outline

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Indiscernability of identicals

Leibniz's law

$$x = y \;\rightarrow\; \forall P\,(P(x) \leftrightarrow P(y))$$

- Reasoning **in logic** is invariant under equality
- **In mathematics**, reasoning should be invariant under weaker notion of sameness!

The equivalence principle

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Reasoning in mathematics should be **invariant under** the appropriate notion of **sameness**.

Notion of sameness depends on the objects under consideration:

- **equal** numbers, functions,...
- **isomorphic** sets, groups, rings,...
- **equivalent** categories
- **biequivalent** bicategories
- ...

Non-examples: statements violating equivalence principle

We can easily **violate** this principle:

Exercise

Find a statement about categories that is not invariant under the equivalence of categories

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A solution

"The category $\mathcal{C}$ has exactly one object."

Maybe this statement is simply silly!

A language for invariant properties

M. Makkai, *Towards a Categorical Foundation of Mathematics*:

> *The basic character of the Principle of Isomorphism is that of a* ***constraint on the language*** *of Abstract Mathematics; a welcome one, since it provides for the separation of sense from nonsense.*

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Goal

to have a **syntactic criterion** for properties and constructions that are invariant under equivalence

How to break the invariance principle for categories...

- Recall: the statement

  The category $\mathcal{C}$ has exactly one object.

  is not invariant under equivalence of categories.
- In general, referring to **equality of objects** breaks invariance, but...

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Problem

"If $\mathsf{source}(g)$ is **equal to** $\mathsf{target}(f)$, then $g \circ f$ exists."

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Problem

"If $\mathsf{source}(g)$ is **equal to** $\mathsf{target}(f)$, then $g \circ f$ exists."

Can we give a definition of category without equality of objects?

. . . and how to fix it.

Solution

Use a logic/language of **dependent types**, in which $s(g) = t(f)$ is encoded by what type of thing f and g are.

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A category consists of

- a collection O of objects
- for each $x, y \in O$, a collection $A(x, y)$ of arrows
- for each $x, y, z \in O$ and each $f \in A(x, y)$ and $g \in A(y, z)$, a composite $g \circ f \in A(x, z)$
- for each $x \in O$, an identity $\mathsf{id}_x \in A(x, x)$
- . . .

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- . . .

Gives rise to **dependently typed language** by adding logical connectors.

Invariance for properties

Theorem (Freyd '76, Blanc '78)

A property of categories (expressed in 2-sorted first order logic) is invariant under equivalence iff it can be expressed in this dependently typed language, using equality for arrows but not for objects.

Invariance for properties

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- What about **constructions** on categories?

Invariance for properties

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- What about **constructions** on categories?
- What about other mathematical structures?

Equivalence principle in the univalent foundations

In the univalent foundations

- an equivalence principle can be proved for a variety of structures
  - sets
  - groups, rings, ...
  - categories
- EP applies not only to properties, but also to constructions: any construction transports suitably along equivalence

Outline

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Univalent foundations

Syntax/Language

- Language of **dependent types**, a.k.a., a type theory
- Logic and constructions are treated uniformly

Semantics/Models

- Natural semantics in **spaces** (simplicial sets)
- Other interpretations are possible

Syntax of UF

Fundamental: **judgment**

$$\text{context} \quad \vdash \quad \text{conclusion}$$

Contexts & judgments

Γ	sequence of variable declarations $x_1 : A_1, x_2 : A_2(x_1), \ldots, x_n : A_n(\vec{x}_i)$
$\Gamma \vdash A$	A is well–formed **type** in context Γ
$\Gamma \vdash a : A$	**term** a is well-formed and of type A
$\Gamma \vdash A \equiv B$	types A and B are **convertible**
$\Gamma \vdash a \equiv b : A$	a is convertible to b in type A

$$(x : \mathsf{Nat}), (f : \mathsf{Nat} \to \mathsf{Nat}) \quad \vdash \quad x^2 : \mathsf{Nat}$$

Rules and derivations

- A **rule** is an implication of judgments,

$$J_1 \wedge J_2 \wedge \ldots \wedge J_n \quad \Longrightarrow \quad J$$

  e.g.,

$$\Gamma \ \vdash \ a \equiv b : A \quad \Longrightarrow \quad \Gamma \ \vdash \ b \equiv a : A$$

  (often written as inference rule with horizontal bar)

- A **derivation** is a sequence of rules.

- Derivations are done **algorithmically**.
  E.g., the judgment $\Gamma \ \vdash \ a \equiv b : A$ is derived automatically (or fails to be derived)

- There is a different equality (**identity**) that can be proved or disproved, see later

- We sometimes omit the context when writing judgments.

About well-typedness

There is no way to derive a judgment of the form

$$(b : \mathsf{Bool}), (f : \mathsf{Nat} \to A) \quad \vdash \quad f(b) : A$$

We are hence not allowed to write $f(b)$ here.

Some differences to set-theoretic membership

- the judgment $a : A$ is **not** a statement that can be proved or disproved
- a valid term has exactly one type up to $\equiv$

Declaring types & terms

Any type (and corresponding term) construction is declared by giving 4 (groups of) rules:

Formation declaring the type

Introduction a way to construct terms of that type

Elimination what can one do with a term of that type

Computation what happens when one does Introduction followed by Elimination

The type of pairs $A \times B$

Formation $(\Gamma \vdash A) \wedge (\Gamma \vdash B) \implies \Gamma \vdash A \times B$

Introduction $(\Gamma \vdash a : A) \wedge (\Gamma \vdash b : B) \implies \Gamma \vdash \mathsf{pair}(a, b) : A \times B$

Elimination $\Gamma \vdash t : A \times B \implies \Gamma \vdash \mathsf{pr}_1(t) : A$
(and analogous for pr_2)

Computation $\mathsf{pr}_1(\mathsf{pair}(a, b)) \equiv a$ and $\mathsf{pr}_2(\mathsf{pair}(a, b)) \equiv b$

The type of functions $A \to B$

Formation $(\Gamma \vdash A) \wedge (\Gamma \vdash B) \implies \Gamma \vdash A \to B$

Introduction $\Gamma, (x : A) \vdash b(x) : B \implies \Gamma \vdash \lambda x.b(x) : A \to B$

Elimination $(\Gamma \vdash f : A \to B) \wedge (\Gamma \vdash a : A) \implies \Gamma \vdash f(a) : B$

Computation $(\lambda x.b)(a) \equiv b[x := a]$

- $\lambda x.b$ corresponds to $x \mapsto b$.
- **Substitution** $b[x := a]$ is built-in
- Example: $\emptyset \vdash \lambda x.x^2 : \mathsf{Nat} \to \mathsf{Nat}$

Type dependency

In particular: dependent type B over A

$$x : A \;\vdash\; B(x)$$

"family B of types indexed by A"

- a type can depend on several variables

The type of dependent functions $\prod_{(x:A)} B$

Formation $(\Gamma \vdash A) \wedge (\Gamma, x : A \vdash B(x)) \implies \Gamma \vdash \prod_{(x:A)} B(x)$

Introduction $\Gamma, (x : A) \vdash b : B \implies \Gamma \vdash \lambda x.b : \prod_{(x:A)} B$

Elimination $(\Gamma \vdash f : \prod_{(x:A)} B) \wedge (\Gamma \vdash a : A) \implies \Gamma \vdash f(a) : B[x := a]$

Computation $(\lambda x.b)(a) \equiv b[x := a]$

- The case $A \to B$ is a special case, where B does not depend on $x : A$

The type of dependent pairs $\sum_{(x:A)} B$

Formation $(\Gamma \vdash A) \wedge (\Gamma, x : A \vdash B(x)) \implies \Gamma \vdash \sum_{(x:A)} B(x)$

Introduction term $\mathsf{pair}(a, b)$ with $a : A$ and $b : B(a)$

Elimination given $t : \sum_{(x:A)} B$, there is $\mathsf{pr}_1(t) : A$ and

$$\mathsf{pr}_2(t) : B(x := \mathsf{pr}_1(t))$$

Computation $\mathsf{pr}_1(a, b) \equiv a$ and $\mathsf{pr}_2(a, b) \equiv b$

- The case $A \times B$ is a special case, where B does not depend on $x : A$

Logic in univalent foundations

Curry-Howard isomorphism:

- propositions are types
- proofs of P are terms of type P

Hence

- In principle, all types could be called propositions
- to prove a propositions P means to construct a term of type P
- In UF, only some types are called "propositions", cf later

The identity type

Formation $(\Gamma \vdash a : A) \wedge (\Gamma \vdash b : A) \implies \Gamma \vdash \mathsf{Id}_A(a, b)$

Introduction $\mathsf{refl}_a : \mathsf{Id}_A(a, a)$

Elimination $C(a, a, \mathsf{refl}_a) \implies C(a, b, p)$

Computation $\ldots$

We also write $a =_A b$ and $a = b$ for $\mathsf{Id}_A(a, b)$

More about identities

Can construct terms of type

- $a = b \to b = a$
- $a = b \times b = c \to a = c$
- $B(a) \times a = b \to B(b)$

Can **not** construct a term UIP of type

$$(x : A), (p : x = x) \;\vdash\; UIP : p =_{(x=x)} \mathsf{refl}_x$$

but can construct a term

$$(x : A), (p : \sum_{y:A} x = y) \;\vdash\; contr : p = \mathsf{pair}(x, \mathsf{refl}_x)$$

Types are ∞-groupoids

Garner, van den Berg

$$(A, \mathsf{Id}_A, \mathsf{Id}_{\mathsf{Id}_A}, \ldots)$$

forms ∞-groupoid, i.e., groupoid laws hold up to "higher" identities

- gives rise to model of type theory in simplicial sets (Voevodsky)
- this model motivates (and justifies) the univalence axiom, cf later

Contractible types, propositions and sets

Definitions:

- A is **contractible** if we can construct a term of type

$$\mathsf{isContr}(A) \stackrel{\text{def}}{=} \sum_{(x:A)} \prod_{(y:A)} y = x$$

- A is a **proposition** if $\prod_{(x\ y:A)} x = y$ is inhabited

- A is a **set** if, for any $x, y : A$, $x = y$ is a proposition

Equivalences

Definition

A map $f : A \to B$ is an **equivalence** if it has contractible fibers, i.e.,

$$\mathsf{isequiv}(f) \stackrel{\text{def}}{=} \prod_{b:B} \mathsf{isContr}\left(\sum_{a:A} f(a) = b\right)$$

The type of equivalences:

$$A \simeq B \stackrel{\text{def}}{=} \sum_{f:A\to B} \mathsf{isequiv}(f)$$

Characterizing some identity types

Can construct equivalences (i.e., terms of type)

- for $\mathsf{pair}(a, b) : A \times B$,

$$\Big(\mathsf{pair}(a, b) = \mathsf{pair}(a', b')\Big) \simeq \Big((a = a') \times (b = b')\Big)$$

- for $f, g : A \to B$

$$\Big(f = g\Big) \simeq \Big(\prod_{a:A} f(a) = g(a)\Big)$$

- ...

Outline

Universes

There is also a type $\mathcal{U}$ that contains all types, i.e., $A : \mathcal{U}$.

- Actually, hierarchy $(\mathcal{U}_i)_{i \in I}$ to avoid paradoxes.
- a dependent type

$$x : A \ \vdash \ B : \mathcal{U}$$

can be considered as a function

$$\lambda x.B : A \to \mathcal{U}$$

Question

What is

$$\mathsf{Id}_{\mathcal{U}}(A, B) \quad ?$$

Voevodsky's Univalence Axiom

Answer 1

$$\mathsf{univalence} : (A =_{\mathcal{U}} B) \simeq (A \simeq B)$$

More controlled:

Answer 2

Define

$$\mathsf{idtoeqv} : \prod_{A,B:\mathcal{U}} (A = B) \to (A \simeq B)$$

$$\mathsf{refl}_A \mapsto \mathsf{id}$$

$$\mathsf{univalence} : \prod_{A,B:\mathcal{U}} \mathsf{isequiv}(\mathsf{idtoeqv}_{A,B})$$

Summary: Univalent Foundations

- A language of **dependent types**, a.k.a. a **type theory**
- With an interpretation in ∞-groupoids (Kan complexes)

Type theory	Interpretation
A type	∞-groupoid A
$a : A$ (term a of type A)	object a of ∞-groupoid A
$f : A \to B$	∞-functor $A \to B$
$p : a =_A b$	path (1-morphism) from a to b in A
$\alpha : p =_{a=_A b} q$	homotopy from p to q in A

- Universe of **sets** given by **discrete groupoids**
- Logic and constructions are treated uniformly in UF

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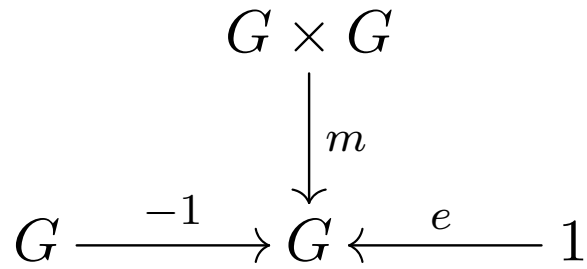
Groups in Univalent Foundations

A **group** in Univalent Foundations is



such that

- G is a **discrete** type, i.e. a "set"
- group axioms are satisfied

Lifting the equivalence principle from types to groups

A **group isomorphism** $G \to G'$ is

- a bijective function on the underlying types $G \to G'$
- compatible with the group structures on G and G'.

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Theorem (EP on types lifts to EP on groups)

- *An isomorphism of groups lifts to an equivalence of all constructions on groups (in UF).*
- *In particular: any statement about groups is invariant under group isomorphism*

Lifting univalence from types to groups

The proof of this statement uses 2 ingredients:

1. $(G = G') \simeq (G \cong G')$
2. Leibniz's law (equality is substitutive)

$(G = G') \simeq (G \cong G')$ is given by the canonical map

$$\mathsf{refl}_G \mapsto \mathsf{id}_G$$

Lifting univalence to algebraic structures (Aczel, Coquand, Danielsson)

For many algebraic structures in univalent foundations, univalence lifts.

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Examples include:

- rings
- posets
- discrete fields
- sets with fixpoint operator

An equivalence principle for categories

Univalence for algebraic structures

For groups, rings, etc., univalence lifts.

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Going to equivalence of categories:

Univalence for categories (A., Kapulkin, Shulman)

For **univalent** categories, equivalence is identity via canonical map

$$(\mathcal{C} = \mathcal{D}) \quad \simeq \quad \mathsf{Equiv}(\mathcal{C}, \mathcal{D}) \quad .$$

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That is, any construction on univalent categories in Univalent Foundations is invariant under **equivalence**.

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That is, any construction on univalent categories in Univalent Foundations is invariant under **equivalence**.

- What is a category in UF?
- What is the **univalence** condition for categories?

Categories in univalent mathematics

A category is

- a type $O : \mathcal{U}$ of objects
- a dependent type $A : O \times O \to \mathcal{S}et$ of arrows
- $\mathsf{id} : \prod_{(a:O)} A(a, a)$
- $(\circ) : \prod_{(a,b,c:O)} A(a, b) \times A(b, c) \to A(a, c)$
- axioms postulating identities of arrows

Categories in univalent mathematics

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- $(\circ) : \prod_{(a,b,c:O)} A(a, b) \times A(b, c) \to A(a, c)$
- axioms postulating identities of arrows
- such that the natural map

$$\mathsf{idtoiso} : \prod_{a,b:O} (a = b) \to \mathsf{iso}(a, b)$$

is an equivalence for any a, b

Some remarks on univalent categories

In simplicial set model

- categories correspond to truncated Segal spaces (Rezk, A model for the homotopy theory of homotopy theory)
- univalence corresponds to completeness

Connection with Freyd, Blanc

Instead of avoiding identity of objects as in [F,B], in a univalent category, identity means (can be replaced with) isomorphism.

Examples of univalent categories

- $\mathcal{S}et$ (discrete types)
- Groups, rings, ... (Structure Identity Principle)
- Functor category $[\mathcal{C}, \mathcal{D}]$, if $\mathcal{D}$ is univalent
- Full subcategories of univalent categories

More examples of univalent categories

- A preorder is univalent iff it is antisymmetric
- If X is of h-level 3, i.e., 1-truncated, then there is a univalent category with X as objects and $\hom(x, y) := (x = y)$
- If $\mathcal{C}$ is univalent, then the category of cones of shape $F : \mathcal{J} \to \mathcal{C}$ is
  - $\rightsquigarrow$ limits (limiting cones) in a univalent category are unique **up to identity**

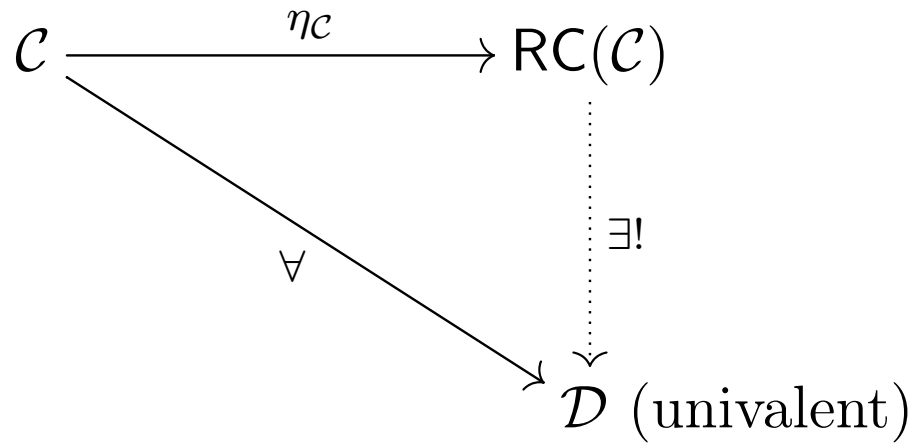
Non-univalent categories

- Any "chaotic" category $\mathcal{C}$ with $\mathcal{C}(x, y) := 1$, for $\mathcal{C}_0$ not a prop

- Any chaotic category $\mathcal{C}$ with an object $c : \mathcal{C}_0$ is **equivalent** to the terminal category $\mathbf{1}$
  - $\rightsquigarrow$ a category can be equivalent to a univalent one without being univalent itself

Rezk completion

To any category $\mathcal{C}$, associate a univalent one, its "Rezk completion",



Intuitively, obtain $\mathsf{RC}(\mathcal{C})_0$ by adding to $\mathcal{C}_0$ as many identities as needed

Construction of the Rezk completion

- $\mathsf{RC}(\mathcal{C})$ is the full image subcategory of the Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{S}et]$
- $\eta_{\mathcal{C}} : \mathcal{C} \to \mathsf{RC}(\mathcal{C})$ is fully faithful and essentially surjective
- precomposition with a ff. and es. functor is ff. and es.
- a ff. and es. functor is an equivalence if source category is univalent
- the object map of an equivalence of univalent categories is an equivalence of types

Some references

- Rezk, *A model for the homotopy theory of homotopy theory*
- Freyd, *Properties invariant within equivalence types of categories*
- Blanc, *Equivalence naturelle et formules logiques en théorie des catégories*
- Coquand, Danielsson, *Isomorphism is equality*
- Kapulkin, Lumsdaine, Voevodsky, *The Simplicial Model of Univalent Foundations*
- Gambino, van den Berg, *Types are weak ω-groupoids*
- Ahrens, Kapulkin, Shulman, *Univalent categories and the Rezk completion* `http://arxiv.org/abs/1303.0584`

Item			
Syntax	Set interpretation	Logic	Simpl. set interpretation
A	set A	proposition A	Kan complex A
$(A, \mathsf{Id}_A, \mathsf{Id}_{\mathsf{Id}_A}, \ldots)$			
$a : A$	$a \in A$	a is a proof of A	$a \in A_0$
$A \times B$	cartesian product	$A \wedge B$	binary product
$A \to B$	set of functions	$A \Rightarrow B$	space of maps
$A + B$	disjoint union	$A \vee B$	binary coproduct
$x : A \ \vdash \ B(x)$	family B of sets indexed by A	predicate B on A	fibration $B \to A$ with fibers $B(x)$
$\sum_{(x:A)} B(x)$	disjoint union	$\exists x, B(x)$	total space of fibration $B \to A$
$\prod_{(x:A)} B(x)$	dependent function	$\forall x, B(x)$	space of sections of fib. $B \to A$
$\mathsf{Id}_A(a, b)$	equality $a = b$		$\hom_A(a, b)$ or $A_1(a, b)$