Univalent Foundations

IV. Univalent categories and SIP

Benedikt Ahrens

Outline

Outline

Indiscernability of identicals

Leibniz's law

$$x = y \quad \rightarrow \quad \forall P\,(P(x) \leftrightarrow P(y))$$

- Reasoning **in logic** is invariant under equality
- **In mathematics**, reasoning should be invariant under weaker notion of sameness!

The equivalence principle

Equivalence principle

Reasoning in mathematics should be **invariant under** the appropriate notion of **sameness**.

Notion of sameness depends on the objects under consideration:

- **equal** numbers, functions,. . .
- **isomorphic** sets, groups, rings,. . .
- **equivalent** categories
- **biequivalent** bicategories
- . . .

Violating the equivalence principle

We can easily **violate** this principle:

Exercise

Find a statement about categories that is not invariant under the equivalence of categories

$$\bullet \rightleftarrows \bullet \quad \simeq \quad \bullet$$

Violating the equivalence principle

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A solution

"The category $\mathscr{C}$ has exactly one object."

Maybe this statement is simply silly!

A language for invariant properties

M. Makkai, *Towards a Categorical Foundation of Mathematics*:

> *The basic character of the Principle of Isomorphism is that of a* ***constraint on the language*** *of Abstract Mathematics; a welcome one, since it provides for the separation of sense from nonsense.*

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Goal

to have a **syntactic criterion** for properties and constructions that are invariant under equivalence

Voevodsky's vision

[. . .] My homotopy lambda calculus is an attempt to create a system which is very good at dealing with equivalences. In particular it is supposed to have the property that given any type expression $F(T)$ depending on a term subexpression t of type T and an equivalence $t \to t'$ (a term of the type $Eq(T; t, t')$) there is a mechanical way to create a new expression F' now depending on t' and an equivalence between $F(T)$ and $F'(T')$ (note that to get F' one can not just substitute t' for t in F – the resulting expression will most likely be syntactically incorrect).
[Email to Daniel R. Grayson, Sept 2006]

Outline

1. The equivalence principle
2. Categories in Univalent Foundations
3. Functors, natural transformations, equivalence of univalent categories
4. Rezk completion

Equivalence principle for groups

We have seen that any property (and construction) on groups in type theory is invariant under group isomorphism.
The proof of this statement uses 2 ingredients:

1. $(\mathbf{G} \rightsquigarrow \mathbf{G}') \quad \simeq \quad (\mathbf{G} \cong \mathbf{G}')$
2. $\mathsf{transport}^T : (\mathbf{G} \rightsquigarrow \mathbf{G}') \times T(\mathbf{G}) \to T(\mathbf{G}')$

The equivalence

$$(\mathbf{G} \rightsquigarrow \mathbf{G}') \quad \simeq \quad (\mathbf{G} \cong \mathbf{G}')$$

is given by the canonical map

$$\mathsf{refl}(\mathbf{G}) \mapsto \mathsf{id}_{\mathbf{G}}$$

Univalence for algebraic structures

Lifting univalence from sets to algebraic structures (Aczel, Coquand, Danielsson)

For many algebraic structures in univalent foundations, paths are isomorphisms.

Examples include:

- monoids, groups, rings
- posets
- discrete fields
- sets with fixpoint operator

This is best studied in terms of **categories**!

Categories in Univalent Foundations — Take I

A naïve definition of categories

A **category** $\mathscr{C}$ is given by

- a type $\mathscr{C}_0$: Type of **objects**
- for any $a, b : \mathscr{C}_0$, a type $\mathscr{C}(a, b)$: Type of **morphisms**
- operations: identity & composition

$$1_a : \mathscr{C}(a, a)$$
$$(\circ)_{a,b,c} : \mathscr{C}(b, c) \to \mathscr{C}(a, b) \to \mathscr{C}(a, c)$$

- axioms: unitality & associativity

$$1 \circ f \rightsquigarrow f \qquad f \circ 1 \rightsquigarrow f \qquad (h \circ g) \circ f \rightsquigarrow h \circ (g \circ f)$$

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Problem: Axioms should be axioms, that is, propositions!

Categories in Univalent Foundations — Take II

A less naïve definition of categories

A category $\mathscr{C}$ is given by

- a type $\mathscr{C}_0$ of objects
- for any $a, b : \mathscr{C}_0$, a **set** $\mathscr{C}(a,b) : \mathsf{Set}$ of morphisms
- operations: identity & composition
- axioms: unitality & associativity

For this definition of category, the axioms are really a property instead of structure.

Isomorphism in a category

Definition (Isomorphism in a category)

A morphism $f : \mathscr{C}(a, b)$ is an **isomorphism** if there are

- $$g : \mathscr{C}(b, a)$$
- $$\eta : g \circ f \rightsquigarrow 1_a \qquad \epsilon : f \circ g \rightsquigarrow 1_b$$

Put differently, we define

$$\mathsf{isiso}(f) \;:\equiv\; \sum_{g : \mathscr{C}(b,a)} \Big((g \circ f \rightsquigarrow 1_a) \times (f \circ g \rightsquigarrow 1_b) \Big)$$

Isomorphism in a category II

Lemma

For any $f : \mathscr{C}(a,b)$, the type $\mathsf{isiso}(f)$ is a proposition. In particular, the inverse is unique if it exists.

Definition (The type of isomorphisms)

$$\mathsf{iso}(a,b) \;:\equiv\; \sum_{f:\mathscr{C}(a,b)} \mathsf{isiso}(f)$$

From paths to isomorphisms

Definition (univalent category)

For a category $\mathscr{C}$ we define

$$\mathsf{idtoiso} : \prod_{a,b:\mathscr{C}_0} (a \rightsquigarrow b) \to \mathsf{iso}(a,b)$$

$$\mathsf{idtoiso}(a,a,\mathsf{refl}(a)) \quad :\equiv \quad 1_a$$

We call the category $\mathscr{C}$ **univalent** if, for any objects $a, b : \mathscr{C}_0$,

$$\mathsf{idtoiso}_{a,b} : (a \rightsquigarrow b) \to \mathsf{iso}(a,b)$$

is an equivalence of types.

Examples of univalent categories

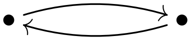
- $\underline{\mathsf{Set}}$
- Groups, rings, . . . (Structure Identity Principle)
- Functor category $[\mathscr{C}, \mathscr{D}]$, if $\mathscr{D}$ is univalent
- Full subcategories of univalent categories

More examples of univalent categories

- A preorder is univalent iff it is antisymmetric
- If X is of h-level 3, then there is a univalent category with X as objects and $\hom(x,y) \quad :\equiv \quad (x \rightsquigarrow y)$
- If $\mathscr{C}$ is univalent, then the category of cones of shape $F : \mathscr{J} \to \mathscr{C}$ is
  - $\rightsquigarrow$ limits (limiting cones) in a univalent category are unique **up to paths**

Non-univalent categories

- Any "chaotic" category $\mathscr{C}$ with $\mathscr{C}(x,y) \simeq \mathbf{1}$, for $\mathscr{C}_0$ not a proposition



- Any chaotic category $\mathscr{C}$ with an object $c : \mathscr{C}_0$ is **equivalent** to the terminal category $\mathbf{1}$
  - ⇝ a category can be equivalent to a univalent one without being univalent itself

Exercises

Lemma

Show $\mathsf{isProp}(\mathsf{isUnivalent}(\mathscr{C}))$.

Lemma

If $\mathscr{C}$ *is univalent, then* $\mathsf{isofhlevel}(3)(\mathscr{C}_0)$.

Outline

What about categories as objects?

Definition (Functor)

A functor F from $\mathscr{C}$ to $\mathscr{D}$ is given by

- a map $F_o : \mathscr{C}_o \to \mathscr{D}_o$
- for any $a, a' : \mathscr{C}_o$, a map $F_{a,a'} : \mathscr{C}(a, a') \to \mathscr{D}(F_o a, F_o a')$
- preserving identity and composition

The category of categories?

- the type of functors from $\mathscr{C}$ to $\mathscr{D}$ does not form a set
- thus there is no category of categories

Natural transformations

Definition (Natural transformation)

Let $F, G : \mathscr{C} \to D$ be functors. A **natural transformation** $\alpha : F \to G$ is given by

- for any $c : \mathscr{C}_0$ a morphism $\alpha_c : \mathscr{D}(Fc, Gc)$ s.t.
- for any $f : \mathscr{C}(c, d)$, a path $Gf \circ \alpha_c \rightsquigarrow \alpha_d \circ Ff$

Exercise: show that the type of natural transformations from F to G forms a set.

Definition (Functor category $[\mathscr{C}, \mathscr{D}]$)

- objects: functors from $\mathscr{C}$ to $\mathscr{D}$
- morphisms from F to G: natural transformations

(Adjoint) equivalence of categories

The functor $F : \mathscr{C} \to \mathscr{D}$ is an **equivalence** if there are

- a functor $G : \mathscr{D} \to \mathscr{C}$
- a natural isomorphism $\eta : 1_{\mathscr{C}} \xrightarrow{\cong} GF$
- a natural isomorphism $\epsilon : FG \xrightarrow{\cong} 1_{\mathscr{D}}$

An equivalence (F, G, η, ϵ) is an adjoint equivalence if F and G form an adjunction.

Theorem

For univalent categories $\mathscr{C}$ and $\mathscr{D}$, the canonical map

$$\mathscr{C} \rightsquigarrow \mathscr{D} \quad \to \quad \mathsf{AdjEquiv}(\mathscr{C}, \mathscr{D})$$

is an equivalence.

Outline

Rezk completion

- *"Being univalent"* is a proposition

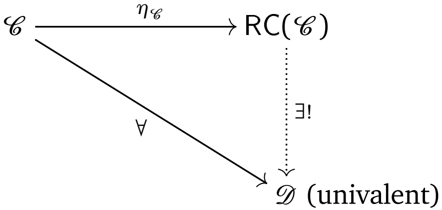
$\Rightarrow$ Forgetful inclusion from univalent categories to categories

Question

Can we associate, to any category, a univalent category in a systematic (a.k.a universal) way?

Universal property of the Rezk completion

To any category $\mathscr{C}$, associate a univalent one, its “Rezk completion”,



Intuitively, obtain $\mathsf{RC}(\mathscr{C})_0$ by adding to $\mathscr{C}_0$ as many paths as there are isomorphisms

Construction of the Rezk completion

Theorem

The inclusion of univalent categories into categories has a left adjoint (in bicategorical sense),

$$\mathscr{C} \mapsto \widehat{\mathscr{C}}, \qquad \textit{\textbf{Rezk completion}} \textit{ of } \mathscr{C} \quad .$$

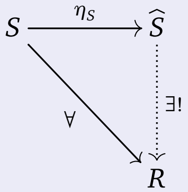
- $\mathsf{RC}(\mathscr{C})$ is the full image subcategory of the Yoneda embedding $\mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathsf{Set}]$
- $\eta_{\mathscr{C}} : \mathscr{C} \to \mathsf{RC}(\mathscr{C})$ is fully faithful and essentially surjective
- precomposition with a ff. and es. functor is ff. and es.
- a ff. and es. functor is an equivalence if source category is univalent
- the object map of an equivalence of univalent categories is an equivalence of types

Special case of Rezk completion: Quotienting

Specialise: category $\rightsquigarrow$ groupoid $\rightsquigarrow$ equivalence relation (setoid)

Theorem

Given a setoid $(S, \sim)$ and a set R, any map $f : S \to R$ such that $s \sim s' \implies f(s) \rightsquigarrow f(s')$ factors uniquely via $\widehat{S}$:



- More direct construction of set-level quotients in univalent foundations by Voevodsky: “type of equivalence classes”

The end

Thanks for staying till the end!