Displayed categories

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Outline

1. Goals and background
2. Displayed category theory
3. Fibrations and comprehension categories
4. Univalence
5. Creation of limits
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Open problem: Initiality conjecture for dependently typed theories

1. Develop notion of ‘signature’ for type theories
2. Construct initial model for any signature

Project with Peter Lumsdaine and Vladimir Voevodsky: *Comparing categorical structures for type theory*

- Categories with families
- Type categories
- Categories with display maps
- Comprehension categories

and formalize results in (univalent) type theory.

*Displayed categories* help with two challenges encountered in this project.
Displayed categories help with two challenges:

**Avoid reasoning about equality of objects of categories**

Equality of objects used in classical formulations of several concepts:
- (Grothendieck) fibrations
- Creation of limits

**Build categories of complex structures step-wise**
- Toy example: category of groups from category of sets + extra structure
- Specifically: mathematical status for extra structure
### Logical setting

<table>
<thead>
<tr>
<th>Type theory with different possible interpretations</th>
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<td><strong>naïve</strong>: types interpreted as sets</td>
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<td><strong>univalent</strong>: types interpreted as simplicial sets</td>
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<table>
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<tr>
<th>Some issues and results trivialize in naïve interpretation</th>
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<tr>
<td>• Transport along equalities</td>
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<tr>
<td>• Results on univalent categories</td>
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Type-theoretical background

- Type theory with $\Sigma$, $\Pi$, $=, o, 1, 2, N, U$
- Type $A$ is **contractible** if has a unique inhabitant
- Type $A$ is a **proposition** if all inhabitants are equal
- Type $A$ is a **set** if all its identity types $a = a'$ are propositions

Results do not rely on univalence or Axiom K
Formalization

- Many of the results formalized, based on the UniMath library
- Available on https://github.com/UniMath/TypeTheory
- Ca. 5000 loc
A category $\mathcal{C}$ is

- a type $\mathcal{C}_o$ of objects
- for any two objects $a, b : \mathcal{C}_o$, a **set** $a \to b$ of arrows
- for any $a : \mathcal{C}_o$, an arrow $1_a : a \to a$
- composition: $(a \to b) \times (b \to c) \to (a \to c)$, denoted $f \cdot g$
- axioms postulating identities of arrows
A **univalent** category $\mathcal{C}$ is

- a type $\mathcal{C}_o$ of objects
- for any two objects $a, b : \mathcal{C}_o$, a **set** $a \to b$ of arrows
- for any $a : \mathcal{C}_o$, an arrow $1_a : a \to a$
- composition: $(a \to b) \times (b \to c) \to (a \to c)$, denoted $f \cdot g$
- axioms postulating identities of arrows
- such that the map

$$\text{idtoiso} : \prod_{a,b : \mathcal{C}_o} ((a = b) \to \text{Iso}(a, b))$$

is an equivalence ‘pointwise’, i.e., for any $a, b : \mathcal{C}_o$,

$$\text{idtoiso}_{a,b} : a = b \sim \text{Iso}(a, b)$$
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Given a category $\mathcal{C}$, a **displayed category** $\mathcal{D}$ *over* $\mathcal{C}$ consists of

- for each $c : \mathcal{C}$, a type $\mathcal{D}_c$
- for each $f : a \to b$ of $\mathcal{C}$ and $x : \mathcal{D}_a$ and $y : \mathcal{D}_b$, a **set** $\text{hom}_f(x,y)$
- for each $c : \mathcal{C}$ and $x : \mathcal{D}_c$, a morphism $1_x : \text{hom}_{1_c}(x,x)$
- for all $f : a \to b$ and $g : b \to c$ in $\mathcal{C}$ and $x : \mathcal{D}_a$ and $y : \mathcal{D}_b$ and $z : \mathcal{D}_c$, a function

$$ (\cdot) : \text{hom}_f(x,y) \times \text{hom}_g(y,z) \to \text{hom}_{f \cdot g}(x,z), $$

- denoted by $(\tilde{f}, \tilde{g}) \mapsto \tilde{f} \cdot \tilde{g} : \text{hom}_{f \cdot g}(x,z)$
- **laws**—well-typed modulo axioms of $\mathcal{C}$
Total category of a displayed category

The total category $\int \mathcal{D}$ of $\mathcal{D}$ over $\mathcal{C}$

- objects are pairs $(a, x)$ where $a : \mathcal{C}$ and $x : \mathcal{D}_a$
- maps $(a, x) \to (b, y)$ are pairs $(f, \tilde{f})$ where $f : a \to b$ and $\tilde{f} : \text{hom}_f(x, y)$

Forgetful functor

$$\pi_1^\mathcal{D} : \int \mathcal{D} \to \mathcal{C}$$

Displayed categories over $\mathcal{C}$ are the same as ‘a category and a functor into $\mathcal{C}$’. 
Examples

The **category of groups** is the total category of the displayed category $\text{grp}$, over set:

- $\text{grp}_X :=$ set of group structures on the set $X$
- for a function $f : X \to Y$ and group structures $(\mu, e)$ on $X$ and $(\mu', e')$ on $Y$,

$$\text{hom}_f((\mu, e), (\mu', e')) :=$$

$f$ is a homomorphism with respect to $(\mu, e), (\mu', e')$

Similarly for **category of topological spaces**.
More examples

- Any category is displayed over 1.
- Given a predicate $P : \mathcal{C}_o \to \text{type}$, setting $\mathcal{D}_c := P(c)$ and $\text{hom}_f(x, y) = 1$ yields

$$\int \mathcal{D} = \text{full subcategory spanned by } P$$

- If every displayed hom-set $\text{hom}_f(x, y)$ of $\mathcal{D}$ is a proposition (inhabited, contractible) then $\pi_1 : \int \mathcal{D} \to \mathcal{C}$ is faithful (full, fully faithful).
- Total category of displayed (co)slice category is arrow category

$$\mathcal{C} \rightarrow \simeq \int_{c : \mathcal{C}} \mathcal{C} / c \simeq \int_{c : \mathcal{C}} c \backslash \mathcal{C}$$

but the $\pi_1$’s are different.
Displayed functors

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor, and $\mathcal{D}$ over $\mathcal{C}$ and $\mathcal{D}'$ over $\mathcal{C}'$. A (displayed) functor $G$ from $\mathcal{D}$ to $\mathcal{D}'$ over $F$ consists of:

- for each $c : \mathcal{C}$, a map
  \[ G_c : \mathcal{D}_c \to \mathcal{D}'_{Fc} \]

- for each $f : c \to c'$ in $\mathcal{C}$, a map
  \[ \text{hom}_f(x, y) \to \text{hom}_{Ff}(Gx, Gy) \]

- dependent analogues of the usual functor laws

Induces total functor $\int G : \int \mathcal{D} \to \int \mathcal{D}'$ commuting with the forgetful functors.
Displayed $X$ over $X$ in the base, inducing $X$ of total categories

For $X$ being

- natural transformations
- adjunctions
- equivalences

In particular,

- displayed category of displayed functors from $\mathcal{D}$ to $\mathcal{D}'$ over category of functors from $\mathcal{C}$ to $\mathcal{C}'$
Fibre categories

Given $\mathcal{D}$ over $\mathcal{C}$ and $c$ an object of $\mathcal{C}$, define fibre category $\mathcal{D}_c$

- $(\mathcal{D}_c)_0 := \mathcal{D}_c$
- $\text{hom}(x,y) := \text{hom}_{1c}(x,y)$

But: displayed $X$ do not generally restrict to $X$ on fibres, requires well-behaved displayed category $\mathcal{D}$
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Fibrations

Definition (cartesian lift, classically)
Given $F : \mathcal{D} \to \mathcal{C}$ and $f : c' \to c$ in $\mathcal{C}$ and $d : \mathcal{D}_0$ such that $Fd = c$, a cartesian lift of $(f, d)$ is an object $d' : \mathcal{D}_0$ with $Fd' = c'$ and a cartesian map $f' : d' \to d$ with $Ff' = f$.

Definition (cartesian lift in terms of displayed categories)
Given $\mathcal{D}$ a displayed category over $\mathcal{C}$ and $f : c' \to c$ in $\mathcal{C}$ and $d : \mathcal{D}_c$, a cartesian lift of $(f, d)$ is an object $d' : \mathcal{D}_c$ and a cartesian map $\bar{f} : \text{hom}_f(d', d)$.

A **fibration** is a displayed category with a cartesian lift for any $f : c' \to c$ and $d : \mathcal{D}_c$. 
Comprehension categories

Definition (comprehension category, classically)

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\chi} & \mathcal{C} \\
\downarrow{p} & & \downarrow{\text{cod}} \\
\mathcal{C} & \rightarrow & \mathcal{C}
\end{array}
\]
commuting strictly

Definition (comprehension category via displayed categories)

• a fibration (in particular, displayed category) \( \mathcal{T} \) over \( \mathcal{C} \)
• a displayed functor \( \mathcal{T} \to \mathcal{C}/- \) over identity functor on \( \mathcal{C} \)

Induces a strictly commuting triangle of functors

\[
\begin{array}{ccc}
\int \mathcal{T} & \xrightarrow{\chi} & \int_{c: \mathcal{C}} \mathcal{C}/c \\
\downarrow{\pi_1} & & \downarrow{\pi_1} \\
\mathcal{C} & \rightarrow & \mathcal{C}
\end{array}
\]
Univalent displayed categories

• Given $\mathcal{D}$ over $\mathcal{C}$ and $i : c \cong c'$ in $\mathcal{C}$, write $\text{Iso}_i(d, d')$ for type of displayed isomorphisms
• For $e : c = c'$ and $d : \mathcal{D}_c$ and $d' : \mathcal{D}_{c'}$,

$$\text{idtoiso}_{e,d,d'} : (d =_e d') \to \text{Iso}_{\text{idtoiso}(e)}(d, d')$$

• Call $\mathcal{D}$ (displayedly) univalent if $\text{idtoiso}_{e,d,d'}$ is an equivalence for all $e, d, d'$.

Lemma

$\mathcal{D}$ displayedly univalent iff all fibre categories $\mathcal{D}_c$ univalent
Structure Identity Principle

Theorem

Given $\mathcal{D}$ over $\mathcal{C}$, if $\mathcal{C}$ is univalent and $\mathcal{D}$ is (displayedly) univalent, then $\int \mathcal{D}$ is univalent.

- Gives a modular way to show that categories of complicated structures are univalent.
- Structure Identity Principle (Aczel, Coquand & Danielsson) is a special case.
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Creation of limits

**Definition (classically)**

A functor $F : \mathcal{A} \to \mathcal{B}$ creates limits of shape $I$ if for any diagram $D : I \to \mathcal{A}$

- for any limit cone $C : B \to FD$ on diagram $FD$ there is a unique cone $C' : A \to D$ such that $F(C') = C$
- $C'$ is limit cone for $D$

**Definition (in terms of displayed categories)**

Let $\mathcal{D}$ be a displayed category over $\mathcal{C}$ and $I$ a category. We say that $\mathcal{D}$ creates limits of shape $I$ if . . .

A displayed category $\mathcal{D}$ over a category $\mathcal{C}$ creates limits (of shape $I$) if and only the functor $\pi_1^\mathcal{D} : \int \mathcal{D} \to \mathcal{C}$ creates limits (of shape $I$) in the classical sense.
Lemma

Suppose the category \( \mathcal{C} \) has limits of shape \( I \), and the displayed category \( \mathcal{D} \) over \( \mathcal{C} \) creates limits of shape \( I \). Then \( \int \mathcal{D} \) has all such limits, and \( \pi_1^{\mathcal{D}} : \int \mathcal{D} \to \mathcal{C} \) preserves them.

Examples

- Given \( F : \mathcal{C} \to \mathcal{C} \), the displayed category of \( F \)-algebras over \( \mathcal{C} \) creates limits. Same for monad algebras.
- The displayed category of groups over sets creates limits.
Future work

• Develop notion of *displayed limit* encompassing and generalizing the creation of limits
• Assemble displayed categories into a *displayed bicategory over the bicategory of categories*
• Displayed categories form a sort of 2-dimensional ‘category with display maps’, with displayed categories over $\mathcal{C}$ being the ‘types in context $\mathcal{C}$’ $\rightsquigarrow$ directed type theory
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• Displayed categories form a sort of 2-dimensional ‘category with display maps’, with displayed categories over $\mathcal{C}$ being the ‘types in context $\mathcal{C}$’ $\rightsquigarrow$ directed type theory

Thanks for your attention!