A higher structure identity principle

Ahrens, North, Shulman, Tsementzis

Categories in type theory

A **category** $\mathscr{C}$ is given by

- a type $\mathscr{C}_0 : \mathscr{U}$ of **objects**
- for any $a, b : \mathscr{C}_0$, a type $\mathscr{C}(a, b) : \mathscr{U}$ of **morphisms**
- operations: identity & composition

$$1_a : \mathscr{C}(a, a)$$
$$(\circ)_{a,b,c} : \mathscr{C}(b, c) \to \mathscr{C}(a, b) \to \mathscr{C}(a, c)$$

- axioms: unitality & associativity

$$1 \circ f = f \qquad f \circ 1 = f \qquad (h \circ g) \circ f = h \circ (g \circ f)$$

Categories in type theory

A **category** $\mathscr{C}$ is given by

- a type $\mathscr{C}_0 : \mathscr{U}$ of **objects**
- for any $a, b : \mathscr{C}_0$, a type $\mathscr{C}(a, b) : \mathscr{U}$ of **morphisms**
- operations: identity & composition

$$1_a : \mathscr{C}(a, a)$$
$$(\circ)_{a,b,c} : \mathscr{C}(b, c) \to \mathscr{C}(a, b) \to \mathscr{C}(a, c)$$

- axioms: unitality & associativity

$$1 \circ f = f \qquad f \circ 1 = f \qquad (h \circ g) \circ f = h \circ (g \circ f)$$

- $\mathscr{C}(a, b)$ should be a set

Categories in type theory

A **category** $\mathscr{C}$ is given by

- a type $\mathscr{C}_0 : \mathscr{U}$ of **objects**
- for any $a, b : \mathscr{C}_0$, a type $\mathscr{C}(a,b) : \mathscr{U}$ of **morphisms**
- operations: identity & composition

$$1_a : \mathscr{C}(a,a)$$
$$(\circ)_{a,b,c} : \mathscr{C}(b,c) \to \mathscr{C}(a,b) \to \mathscr{C}(a,c)$$

- axioms: unitality & associativity

$$1 \circ f = f \qquad f \circ 1 = f \qquad (h \circ g) \circ f = h \circ (g \circ f)$$

- $\mathscr{C}(a,b)$ should be a set
- Univalence: $a = b \to \mathsf{iso}(a,b)$ should be an equivalence

Local univalence implies global univalence

Theorem

For precategories A and B, let $A \simeq B$ denote the type of equivalence functors from A to B. Then if A and B are univalent categories, we have

$$(A =_{\mathsf{UCat}} B) = (A \simeq B).$$

Goal

Prove a similar theorem for other categorical structures.

Envisioned result

Given a signature $\mathscr{L}$, and two $\mathscr{L}$-univalent $\mathscr{L}$-structures M and N, then

$$(M = N) = (M \simeq_{\mathscr{L}} N)$$

Need notions of

- signature
- structure for a signature
- equivalence of structures
- $\mathscr{L}$-isomorphism and $\mathscr{L}$-univalence (a.k.a. saturation)

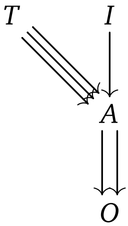
Outline

1. Example: FOLDS categories
2. FOLDS in 2-level type theory

Makkai's FOLDS

- Signature := inverse category

$\mathscr{L}_{\mathrm{cat}} :=$



- Structures for a signature $\mathscr{L} := \mathscr{L} \to \mathbf{Set}$
- Signature only specifies the data, e.g., of categories
- FOLDS language for axioms

FOLDS structures in type theory

In type theory natural to define a structure for $\mathscr{L}_{\mathrm{cat}}$ to be

- A type $MO : \mathscr{U}$;
- A family $MA : MO \times MO \to \mathscr{U}$;
- A family $MI : \prod_{(x:MO)} MA(x,x) \to \mathscr{U}$; and
- A family
  $MT : \prod_{(x,y,z:MO)} MA(x,y) \to MA(y,z) \to MA(x,z) \to \mathscr{U}$.

Here

- MI, MT predicates "being an identity", "being a (the?) composite"
- MI, MT being pointwise propositions is not enforced (will pop out of general univalence condition)
- Same for MA being pw. sets

FOLDS language by example

$$\forall(x,y,z:O).\forall(f:A(x,y)).\forall(g:A(y,z)).\exists(h:A(x,z)).T(x,y,z,f,g,h)$$

FOLDS with equality

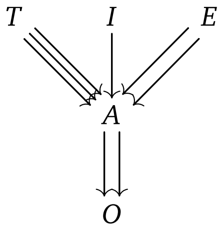
$$\forall(x,y,z:O).\forall(f:A(x,y)).\forall(g:A(y,z)).\forall(h,h':A(x,z)).$$
$$T(x,y,z,f,g,h)\to T(x,y,z,f,g,h')\to(h=h')$$

- Equality only allowed for sorts one below top-level — e.g., not allowed on objects!
- Equality only between “parallel” elements

FOLDS without equality

Add an equality predicate in the signature

$\mathscr{L}_{\text{cat+E}} :=$



$$\forall(x,y,z : O).\forall(f : A(x,y)).\forall(g : A(y,z)).\forall(h,h' : A(x,z)).$$
$$T(x,y,z,f,g,h) \to T(x,y,z,f,g,h') \to E(h,h')$$

- Need axioms stating that E is equivalence relation and congruence
- For univalent structures, $E(f,g) \leftrightarrow (f = g)$ will pop out of general univalence condition

1-saturated FOLDS-categories

A 1-saturated FOLDS-category consists of

- $MO : \mathscr{U}$;
- $MA : MO \times MO \to \mathscr{U}$;
- $MI : \prod_{(x:MO)} MA(x,x) \to \mathscr{U}$;
- $MT : \prod_{(x,y,z:MO)} MA(x,y) \to MA(y,z) \to MA(x,z) \to \mathscr{U}$
- $ME : \prod_{(x,y:MO)} MA(x,y) \to MA(x,y) \to \mathscr{U}$,

such that

- $MI_x(f)$, $MT_{x,y,z}(f,g,h)$, and $ME_{x,y}(f,g)$ are propositions
- $MA(x,y)$ is a set,
- $ME_{x,y}(f,g) \leftrightarrow (f = g)$,

and the axioms of a category are satisfied.

Lemma

The type of 1-sat. FOLDS-cats is equivalent to the type of precategories.

FOLDS isomorphism for categories

Goal

Define the type of isomorphisms $a \cong b$ without referring to categorical axioms, using only the FOLDS structure.

Using Yoneda:

- For each $x : O$, an isomorphism $\phi_{x\bullet} : A(x,a) \cong A(x,b)$.
- For each $x,y : O$, $f : A(x,y)$, $g : A(y,a)$, and $h : A(x,a)$, we have

$$T_{x,y,a}(f,g,h) \Leftrightarrow T_{x,y,b}(f,\phi_{y\bullet}(g),\phi_{x\bullet}(h))$$

$(\phi_{y\bullet}(g) \circ f = \phi_{x\bullet}(g \circ f))$

Problem: not symmetric, not algorithmic

FOLDS isomorphism for categories

Equivalences between hom-sets with a and b substituted into *all* possible "collections of holes":

1. For each $x : O$, an isomorphism $\phi_{x\bullet} : A(x,a) \cong A(x,b)$.
2. For each $z : O$, an isomorphism $\phi_{\bullet z} : A(a,z) \cong A(b,z)$.
3. An isomorphism $\phi_{\bullet\bullet} : A(a,a) \cong A(b,b)$.

and logical equivalences between all "relations with holes":

$$T_{x,y,a}(f,g,h) \Leftrightarrow T_{x,y,b}(f,\phi_{y\bullet}(g),\phi_{x\bullet}(h)) \tag{1}$$

$$T_{x,a,z}(f,g,h) \Leftrightarrow T_{x,b,z}(\phi_{x\bullet}(f),\phi_{\bullet z}(g),h) \tag{2}$$

$$T_{a,z,w}(f,g,h) \Leftrightarrow T_{b,z,w}(\phi_{\bullet z}(f),g,\phi_{\bullet w}(h)) \tag{3}$$

$$T_{x,a,a}(f,g,h) \Leftrightarrow T_{x,b,b}(\phi_{x\bullet}(f),\phi_{\bullet\bullet}(g),\phi_{x\bullet}(h)) \tag{4}$$

$$T_{a,x,a}(f,g,h) \Leftrightarrow T_{b,x,b}(\phi_{\bullet x}(f),\phi_{x\bullet}(g),\phi_{\bullet\bullet}(h)) \tag{5}$$

$$T_{a,a,x}(f,g,h) \Leftrightarrow T_{b,b,x}(\phi_{\bullet\bullet}(f),\phi_{\bullet x}(g),\phi_{\bullet x}(h)) \tag{6}$$

$$T_{a,a,a}(f,g,h) \Leftrightarrow T_{b,b,b}(\phi_{\bullet\bullet}(f),\phi_{\bullet\bullet}(g),\phi_{\bullet\bullet}(h)) \tag{7}$$

$$I_{a,a}(f) \Leftrightarrow I_{b,b}(\phi_{\bullet\bullet}(f)) \tag{8}$$

Univalent FOLDS categories

Theorem

In any 1-sat. FOLDS cat, the type of FOLDS-isomorphisms from a to b is equivalent to the type of ordinary isomorphisms $a \cong b$.

Definition

A 0-sat. FOLDS cat is a 1-sat. FOLDS cat such that for all $a, b : MO$, the canonical map from $(a = b)$ to the type $(a \cong b)$ of FOLDS-isomorphisms is an equivalence.

Theorem

A 1-sat. FOLDS cat is 0-sat. if and only if its corresponding precategory is a univalent category.

Reminder: 1-saturated FOLDS-categories

A 1-saturated FOLDS-category consists of

- $MO : \mathscr{U}$;
- $MA : MO \times MO \to \mathscr{U}$;
- $MI : \prod_{(x:MO)} MA(x,x) \to \mathscr{U}$;
- $MT : \prod_{(x,y,z:MO)} MA(x,y) \to MA(y,z) \to MA(x,z) \to \mathscr{U}$
- $ME : \prod_{(x,y:MO)} MA(x,y) \to MA(x,y) \to \mathscr{U}$,
- $MI_x(f)$, $MT_{x,y,z}(f,g,h)$, and $ME_{x,y}(f,g)$ are propositions
- $MA(x,y)$ is a set,
- $ME_{x,y}(f,g) \leftrightarrow (f = g)$,

and the axioms of a category are satisfied.

- ad hoc assumptions on $MA(x,y)$ being a set and "standardness"
- Univalence condition was only imposed on objects of MO

Replacing ad hoc assumptions by A-univalence

A 2-saturated FOLDS cat consists of

- A type $MO : \mathcal{U}$;
- A family $MA : MO \times MO \to \mathcal{U}$;
- A family $MI : \prod_{(x:MO)} MA(x,x) \to \mathcal{U}$;
- A family $MT : \prod_{(x,y,z:MO)} MA(x,y) \to MA(y,z) \to MA(x,z) \to \mathcal{U}$; and
- A family $ME : \prod_{(x,y:MO)} MA(x,y) \to MA(x,y) \to \mathcal{U}$,

such that

- Each type $MI_x(f)$, $MT_{x,y,z}(f,g,h)$, and $ME_{x,y}(f,g)$ is a proposition;
- categorical axioms expressed modulo ME, with ME assumed eq. rel. and cong.

Univalence at level A for 2-saturated FOLDS cats

Given $f, g : A(a,b)$ in a 2-saturated FOLDS cat.
An iso $i : f \cong g$ given by logical equivalences between T, I, and E with occurrences of f replaced by g in all possible ways.

$$
\begin{aligned}
T_{x,a,b}(u,f,v) &\Leftrightarrow T_{x,a,b}(u,g,v) \\
T_{a,x,b}(u,v,f) &\Leftrightarrow T_{a,x,b}(u,v,g) \\
T_{a,b,x}(f,u,v) &\Leftrightarrow T_{x,a,b}(g,u,v) \\
E_{a,b}(f,u) &\Leftrightarrow E_{a,b}(g,u) \\
E_{a,b}(u,f) &\Leftrightarrow E_{a,b}(u,g) \\
&\vdots
\end{aligned}
$$

A-isomorphisms

Theorem

A 2-saturated FOLDS cat is 1-saturated if and only if the map $(f = g) \to (f \cong g)$ is an equivalence for all f, g.

Hence, "saturation at A" is equivalent to "$A(x, y)$ is a set and standardness"

A-isomorphisms

Theorem

A 2-saturated FOLDS cat is 1-saturated if and only if the map $(f = g) \to (f \cong g)$ is an equivalence for all f, g.

Hence, "saturation at A" is equivalent to "$A(x, y)$ is a set and standardness"

Next goal

Replace assumptions of MI, MT, ME being pw propositions by some sort of univalence condition

FOLDS categories

Definition

A **FOLDS-category** consists of the same data and axioms as a 2-saturated, but without T, I, and E being propositions.

What is $t \cong t'$ in T? Contractible, since nothing "above T".

Theorem

A FOLDS-category is 2-saturated if and only if the canonical maps

$$(t = t') \to (t \cong t')$$
$$(i = i') \to (i \cong i')$$
$$(e = e') \to (e \cong e')$$

are equivalences for all inhabitants of T, I, and E, respectively.

Conclusions from this example

- Notion of FOLDS category only requires $\mathscr{L}_{\mathrm{cat+E}}$ and corresponding axioms
- Notion of isomorphism and saturation conditions are independent of axioms of a FOLDS category
- (Notion of equivalence of structures does not depend on axioms)

Outline

1. Example: FOLDS categories
2. FOLDS in 2-level type theory

2-Level Type Theory

2-Level Type Theory (2LTT) introduced by Capriotti with collaborators

- **fibrant** fragment which consists of $\prod$, Σ, $+$, 1, 0, $\mathbf{N}$, intensional $=$, and a hierarchy of univalent universes $\mathcal{U}$.
- **strict** fragment which consists of $+^s, 0^s, \mathbf{N}^s$, a strict equality $\equiv$ that satisfies UIP and function extensionality, a hierarchy of strict universes $\mathcal{U}^s$, and which shares the the type constructors $\prod, \Sigma, 1$ with the fibrant fragment.

The other level of 2LTT is the The types of the strict fragment are called *pretypes* and the types of the fibrant fragment are called *fibrant* types.

Pretype categories

Definition

An s-**category** $\mathscr{C}$ is given by the following data:

1. A pretype $\mathscr{C}_0$ of *objects* (also often denoted $\mathscr{C}$).
2. For each $x, y : \mathscr{C}$ a pretype $\mathscr{C}(x, y)$ of *arrows*.
3. For each $x : \mathscr{C}$ an arrow $1 : \mathscr{C}(x, x)$.
4. A *composition* operation $\circ : \mathscr{C}(y, z) \to \mathscr{C}(x, y) \to \mathscr{C}(x, z)$ that is strictly associative and for which 1_x is a strict left and right unit.

Signatures

Definition

An **inverse category** is a pair $\langle \mathscr{C}, \rho \rangle$ where $\mathscr{C}$ is an *s*-category and $\rho : \mathscr{C} \to \omega^{op}$ is a *rank* functor which reflects identities in the sense that

$$\prod_{(x,y : \mathscr{C})} \prod_{(f : \mathscr{C}(x,y))} \rho(x) \equiv \rho(y) \to \left(\sum_{p : x \equiv y} (p_*(f) \equiv 1_y) \right) \tag{9}$$

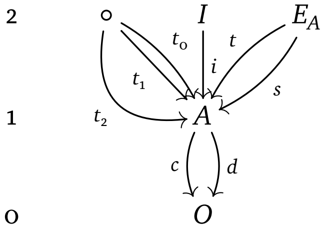
An **inverse category of finite height** consists of an inverse category $\langle \mathscr{C}, \rho \rangle$ and a natural number $n : \omega$ such that every object has rank strictly less than n.

Definition (Signature)

A **signature** is an inverse category $\mathscr{L}$ with some extra conditions. We refer to the objects of a signature as **sorts**.

Signature of categories

By $\mathscr{L}_{\text{cat}}$ we will denote the following FOLDS signature



subject to some relations, e.g.,

$$dt_0 \equiv dt_2, ct_1 \equiv ct_2, dt_1 \equiv ct_0$$

$$ds \equiv dt, cs \equiv ct$$

Derivative of a signature wrt a map on level 0

Definition ($\mathcal{L}'_{M_0}$)

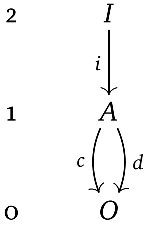
Let $\mathcal{L}$ be a signature and $M_0 : \mathcal{L}_0 \to \mathcal{U}$ a fibrant type family indexed by the pretype $\mathcal{L}_0$ of sorts of $\mathcal{L}$ of rank 0. The **derivative of $\mathcal{L}$ with respect to** M_0 is an s-category $\mathcal{L}'_{M_0}$ defined as follows:

- $\mathrm{ob}(\mathcal{L}'_{M_0}) =_{\mathrm{df}} \sum_{(K:\mathcal{L}_{>0})} \prod_{(K_g:\mathcal{L}_0)} \prod_{(g:K\to K_g)} M_0 K_g$
- $\mathcal{L}'_{M_0}(\langle I, \mathbf{a}\rangle, \langle A, \mathbf{b}\rangle) =_{\mathrm{df}} \sum_{(h:\mathcal{L}(I,A))} \prod_{(K_g:\mathcal{L}_0)} \prod_{(g:\mathcal{L}(A,K_g))} a_{gh} \equiv b_g$

where $\mathbf{a} =_{\mathrm{df}} (a_g : M_0 K_g)_{g:K\to K_g}$ and similarly for $\mathbf{b}$. Composition and identities in $\mathcal{L}'_{M_0}$ are inherited from $\mathcal{L}$. We define a rank function ρ' for $\mathcal{L}'_{M_0}$ by

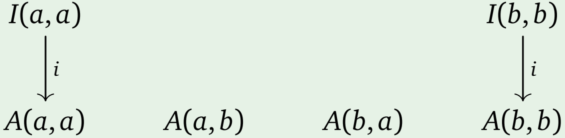
$$\rho'(\langle K, \mathbf{a}\rangle) =_{\mathrm{df}} \rho(K) - 1 \tag{10}$$

Example of derivation



Example

In $\mathscr{L}_{\mathrm{rg}}$ we have $(\mathscr{L}_{\mathrm{rg}})_0 \equiv \{O\}$. Let M_0 be a (a function picking out) the two-element set $\{a, b\}$. Then $(\mathscr{L}_{\mathrm{rg}})'_{M_0}$ is the following signature, with four sorts of rank 0 and two sorts of rank 1:



Structures

Definition ($\mathscr{L}$-structure)

An $\mathscr{L}$**-structure** M is defined by induction on the height of $\mathscr{L}$.

1. If $\mathscr{L}$ has height 0, then there is a unique $\mathscr{L}$-structure.
2. If $\mathscr{L}$ has height $n+1$, then an $\mathscr{L}$-structure is a pair $\langle M_0, M' \rangle$, where $M_0 : \mathscr{L}_0 \to \mathscr{U}$ is a function and M' is an $\mathscr{L}'_{M_0}$-structure. Note that the induction is well-founded since $\mathscr{L}'_{M_0}$ has height n.

Morphism of $\mathscr{L}$-structures

Definition (Morphism of $\mathscr{L}$-structures)

Let M, N be $\mathscr{L}$-structures. A **morphism of $\mathscr{L}$-structures** $f : M \to N$ is defined inductively as follows.

1. If $\mathscr{L}$ has height 0, there is a unique morphism from the unique structure to itself
2. If $\mathscr{L}$ has height $n+1$, then f consists of the following data:
   - A map $f_0 : \prod_{(K:\mathscr{L}_0)} M_0 K \to N_0 K$
   - A morphism of $\mathscr{L}$-structures $f' : M' \to (Id'_{f_0})^* N'$, where $Id'_{f_0} : \mathscr{L}'_M \to \mathscr{L}'_N$ is the discrete opfibration induced by f_0 and the identity functor of $\mathscr{L}$.

Equivalence of $\mathscr{L}$-structures

Definition (Levelwise Equivalence)

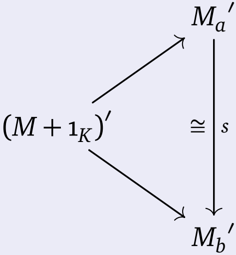
We define what it means for a morphism $f : M \to N$ of $\mathscr{L}$-structures to be a **levelwise equivalence** by induction on height.

1. The unique map between empty structures is a levelwise equivalence.
2. For positive height, f is a levelwise equivalence if $f_0 : \prod_{(K:\mathscr{L}_0)} M_0 K \to N_0 K$ is a family of equivalences of types and $f' : M' \to (Id'_{f_0})^* N'$ is a levelwise equivalence.

$\mathscr{L}$-isomorphism

(Pseudo-)definition

Let M be an $\mathscr{L}$-structure, $K : \mathscr{L}_0$, and $a, b : MK$. The type $a \cong_M b$ of $\mathscr{L}$-isomorphisms from a to b is defined to be the type of levelwise equivalences $s : M_a \cong M_b{}'$ in $\mathbf{Struc}_{\mathscr{L}'_{M_0+1_K}}$ under $(M + 1_K)'$, i.e. those s such that the following triangle commutes:



Remains to be done: define $M + 1_K$ without using excluded middle.

$\mathscr{L}$-univalence

Definition ($\mathscr{L}$-univalence)

Let K be an object of $\mathscr{L}_{\mathrm{o}}$. We say that M is **K-univalent** if

$$\mathbf{Sat}(M,K) =_{\mathrm{df}} \prod_{x,y\,:\,MK} \mathsf{isequiv}(\mathsf{idtoiso}^{MK}_{x,y}) \tag{11}$$

is inhabited. We say that M is **univalent** if

$$\mathbf{Sat}(M) =_{\mathrm{df}} \mathbf{Sat}(M') \times \prod_{K:\mathscr{L}_{\mathrm{o}}} \prod_{x,y\,:\,MK} \mathsf{isequiv}(\mathsf{idtoiso}^{MK}_{x,y}) \tag{12}$$

is inhabited. We write $\mathbf{SatStr}_{\mathscr{L}}$ for the type of (totally) univalent $\mathscr{L}$-structures.

Envisioned results

Envisioned result I

Let $\mathscr{L}$ be of height n and $K : \mathscr{L}_0$. Let M be a univalent $\mathscr{L}$-structure. Then $M_0 K$ is of h-level n.

Envisioned result II

The type $\boldsymbol{SatStr}_{\mathscr{L}}$ is of h-level $h(\mathscr{L}) + 1$.

Envisioned result III

Consider $\mathscr{L}$-structures M, N for some signature $\mathscr{L}$ such that M is univalent. Then

$$\mathbf{VSS}(M, N) \simeq (M = N) \tag{13}$$