Univalent Foundations and the equivalence principle

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Outline

1. The equivalence principle
2. The equivalence principle in Univalent Foundations

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Reasoning in mathematics should be **invariant under** the appropriate notion of **equivalence**.

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Notion of equivalence depends on the objects under consideration:

- **equal** numbers, functions,. . .
- **isomorphic** sets, groups, rings,. . .
- **equivalent** categories
- **biequivalent** bicategories
- . . .

Non-examples: statements violating equivalence principle

We can easily **violate** this principle:

Exercise

Find a statement about categories that is not invariant under the equivalence of categories

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A solution

“The category $\mathcal{C}$ has exactly one object.”

Maybe this statement is simply silly!

A language for invariant properties

M. Makkai, *Towards a Categorical Foundation of Mathematics*:

> *The basic character of the Principle of Isomorphism is that of a* ***constraint on the language*** *of Abstract Mathematics; a welcome one, since it provides for the separation of sense from nonsense.*

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Goal

to have a **syntactic criterion** for properties and constructions that are invariant under equivalence

How to break the invariance principle for categories. . .

- Recall: the statement

  The category $\mathcal{C}$ has exactly one object.

  is not invariant under equivalence of categories.
- In general, referring to **equality of objects** breaks invariance, but. . .

How to break the invariance principle for categories...

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- even the **definition** of category refers to equality of objects:

Problem

"If $\mathsf{source}(g)$ is **equal to** $\mathsf{target}(f)$, then $g \circ f$ exists."

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Problem

"If $\mathsf{source}(g)$ is **equal to** $\mathsf{target}(f)$, then $g \circ f$ exists."

Can we give a definition of category without using equality of objects?

...and how to fix it.

Solution

Use a logic/language of **dependent types**, in which $\mathsf{source}(g) = \mathsf{target}(f)$ is encoded by what type of thing f and g are.

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A category consists of

- a collection O of objects
- for each $x, y \in O$, a collection $A(x, y)$ of arrows
- for each $x, y, z \in O$ and each $f \in A(x, y)$ and $g \in A(y, z)$, a composite $g \circ f \in A(x, z)$
- for each $x \in O$, an identity $\mathsf{id}_x \in A(x, x)$
- ...

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Gives rise to **dependently typed language** by adding logical connectors.

Invariance for statements

Theorem (Freyd '76, Blanc '78)

A property of categories (expressed in 2-sorted first order logic) is invariant under equivalence iff it can be expressed in this dependently typed language, using equality for arrows but not for objects.

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- What about **constructions** on categories?

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- What about **constructions** on categories?
- What about other mathematical structures?

Outline

Univalent Foundations

- A language of **dependent types**, a.k.a. a **type theory**
- With an interpretation in ∞-groupoids (i.e. Kan complexes)

Type theory	Interpretation
type A	∞-groupoid A
term a of type A	object a of ∞-groupoid A
function $A \to B$	∞-functor $A \to B$

- Universe of **sets** given by **discrete ∞-groupoids**
- Properties and constructions are treated uniformly in UF

The Univalence Axiom

Univalence Axiom (Voevodsky): EP for types

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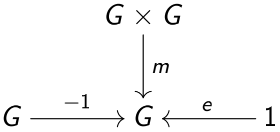
Definition

A map $f : A \to B$ of types is an **equivalence** if there is $g : B \to A$ such that

- for any $a : A, g(f(a)) \simeq a$
- for any $b : B, f(g(b)) \simeq b$

A **group** in Univalent Foundations is



such that

- G is a **discrete** type, i.e. a "set"
- group axioms are satisfied

Lifting the equivalence principle to algebraic structures

A **group isomorphism** $G \to G'$ is

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- compatible with the group structures on G and G'.

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EP on types lifts to EP on groups:

Structure Identity Principle (Aczel, Coquand, Danielsson)

- An iso of groups lifts to an equivalence of all constructions on groups (in UF).
- In particular: any statement about groups is invariant under group iso,
- and similarly for other algebraic structures.

An equivalence principle for categories

Express Structure Identity Principle differently:

SIP categorically:

In the categories of sets, groups, rings,. . . , any construction expressible in UF is invariant under **isomorphism**.

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Going to equivalence of categories:

Theorem (A., Kapulkin, Shulman)

In the bicategory of ***saturated*** *categories, any construction in Univalent Foundations is invariant under* ***equivalence****.*

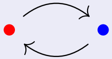
What is this **saturation** condition?

Saturation

Saturation, intuitively

In a saturated category, isomorphic objects are indistinguishible, i.e., they satisfy the same properties.

Non-example



Examples of saturated categories

- Set
- Grp, Rng,... (categories of algebraic structures)
- $[\mathcal{C}, \mathcal{D}]$, if $\mathcal{D}$ is saturated
- any full subcategory of a saturated category

Goals

- Generalize saturation condition to arbitrary (higher-categorical) structures given by a **signature**
- Prove EP for saturated such structures

References

- A., Kapulkin, Shulman: *Univalent categories and the Rezk completion*
- Blanc: *Équivalence naturelle et formules logiques en théorie des catégories*
- Coquand, Danielsson: *Isomorphism is equality*
- Freyd: *Properties invariant within equivalence types of categories*
- Makkai: *Towards a Categorical Foundation of Mathematics*
- Rezk: *A model for the homotopy theory of homotopy theory*
- Voevodsky: *Univalent Foundations project*