# Category theory in the Univalent Foundations

Benedikt Ahrens

joint work with Krzysztof Kapulkin and Michael Shulman

Séminaire LDP, I2M, Marseille

# Univalent Foundations

## Univalent Foundations a.k.a. Homotopy Type Theory

- is type theory with a semantics in spaces
- comes with an additional axiom compared to MLTT
- provides a **synthetic** way to do homotopy theory

## Most importantly (for me)

Univalent Foundations captures

**reasoning modulo “indistinguishability”**.

# Motivation: equality = indistinguishability

## In type theory, equal objects $t = t'$ are indistinguishable

- we cannot define a predicate $P$ such that $P(t)$ and not $P(t')$
- ensured by substitution principle

$$\mathsf{subst} : (t = t') \times P(t) \to P(t')$$

## Conversely, are indistinguishable objects equal in type theory?

- **no generic internal notion of indistinguishability**
- for some types we have an intuition about what **should** be indistinguishable

# Indistinguishability for functions and types

### When are two functions indistinguishable?

$\rightsquigarrow$ when they are indistinguishable on any input!

- “indistinguishability = equality” requires axiom of **functional extensionality**

### When are two types indistinguishable?

$\rightsquigarrow$ when they are isomorphic!

- “indistinguishability = equality” requires **univalence** axiom

# About indistinguishable categories

## In this talk

define a notion of **category** in type theory for which

$$\text{indistinguishability} = \text{equality}$$

When are two categories $\mathcal{C}$ and $\mathcal{D}$ indistinguishable?

| | |
|---|---|
| $f = g$ | $\forall\, x, fx = gx$ |
| $A = B$ | $A \simeq B$ |
| $\mathcal{C} = \mathcal{D}$ | ??? |

# 3 kinds of sameness for categories

| | |
|---|---|
| **Equality** | $\mathcal{C} = \mathcal{D}$ |
| **Isomorphism** | $\mathcal{C} \cong \mathcal{D}$ |
| **Equivalence** | $\mathcal{C} \simeq \mathcal{D}$ |

- most properties of categories invariant under **equivalence**
- we can only substitute **equals for equals**
- in set-theoretic foundations these notions are worlds apart

**In this talk:**

Define categories in the **Univalent Foundations** for which all three coincide

# Outline

1. Introduction to Univalent Foundations
   - Type theory and its homotopy interpretation
   - Logic in type theory: homotopy levels
   - The Univalence Axiom

2. Category Theory in Univalent Foundations
   - Categories: basic definitions
   - Univalent categories: definition & some properties
   - The Rezk completion

# Table of Contents

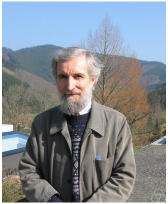
# Univalent Foundations

## What are the Univalent Foundations?

- Intensional Martin-Löf Type Theory

$\rightsquigarrow$ *Types as Spaces* interpretation, i.e. Homotopy Type Theory

$+$ Voevodsky's **Univalence Axiom**



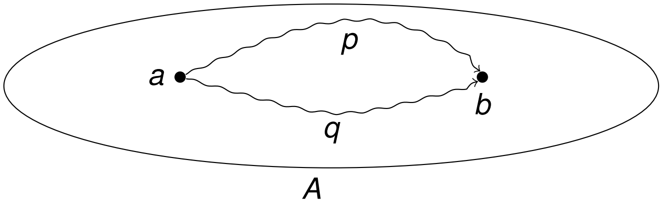
# Martin-Löf TT and its Homotopy Interpretation

| Type theory | Notation | Interpretation |
|---|---|---|
| Inhabitant | $a : A$ | $a$ is a point in space $A$ |
| Dependent type | $x : A \vdash B(x)$ | fibration $\sum_{(x:A)} B(x) \to A$ |
| Sigma type | $\sum_{x:A} B(x)$ | total space of a fibration |
| Product type | $\prod_{x:A} B(x)$ | space of sections of a fibration |
| Coproduct type | $A + B$ | disjoint union |
| Identity type | $\mathsf{Id}_A(a, b)$ | space of **paths** $p : a \rightsquigarrow b$ |

- other types as needed (type $\mathbf{N}$ of naturals, empty type)

# Interpretation: identity type as path space

- For two terms $a, b : A$ of a type $A$, there is a type $\mathsf{Id}(a, b)$
- terms $p, q : \mathsf{Id}(a, b)$ are interpreted as paths $p, q : a \rightsquigarrow b$




### Mixing syntax and semantics

- Call a term $p : \mathsf{Id}(a, b)$ a "path from $a$ to $b$", write $p : a \rightsquigarrow b$
- Say $a$ and $b$ are **homotopic** if there is a path $p : a \rightsquigarrow b$.

# The homotopy interpretation of identity types

Interpretation of the operations on paths:

| Type theory | Interpretation | Notation |
|---|---|---|
| refl | constant path on $a$ | $\mathsf{refl}(a)$ |
| inverse | path reversal | $p^{-1}$ |
| concat | path concatenation | $p \star q$ |
| higher identity type | paths between paths | $p \rightsquigarrow q$ |
| | "continuous deformations" | |

# Table of Contents

# Curry-Howard: propositions as some types

**Definition (Proposition in UF)**

A type $A$ is a **proposition** if all its inhabitants are homotopic, ie. if one can construct a term of type

$$\mathsf{isProp}(A) := \prod_{x:A} \prod_{y:A} \mathsf{Id}_A(x, y) \quad .$$

- "Being a proposition" is a proposition, ie. one can prove

$$\mathsf{isProp}(\mathsf{isProp}(A))$$

- Intuitively, a proposition is either empty or a singleton.

# Quantification in UF

**$\forall x : A.P(x)$**

$\prod_{x:A} P(x)$ is a proposition if $P(x)$ is a proposition for any $x$

# Quantification in UF

### $\forall x : A.P(x)$

$\prod_{x:A} P(x)$ is a proposition if $P(x)$ is a proposition for any $x$

### $\exists x : A.P(x)$

$\sum_{x:A} P(x)$ is **not** a proposition even if $P(x)$ is for any $x$

- Example: $\sum_{n:Nat} \mathsf{even}(n)$
- **Truncation** necessary to obtain a proposition

# Sets in Univalent Foundations

**Definition (Sets)**

Type $A$ is a **set** if the type $\mathsf{Id}_A(x, y)$ is a proposition for any $x, y$

$$\mathsf{isSet}(A) := \prod_{x\ y:A} \mathsf{isProp}(\mathsf{Id}(x, y))$$

- Points of a set are equal in a unique way, if they are.
- Sets are precisely those types satisfying UIP / Axiom K.
- Sets correspond to **discrete spaces**.

# About the use of the word "unique"

## Definition

We call the point $a : A$ **unique** if any point $x : A$ is **homotopic** to $a$, ie. if we can construct a term of type

$$\prod_{x:A} \mathsf{Id}(x, a)$$

# About the use of the word "unique"

**Definition**

We call the point $a : A$ **unique** if any point $x : A$ is **homotopic** to $a$, ie. if we can construct a term of type

$$\prod_{x:A} \mathsf{Id}(x, a)$$

A type $A$ with a unique point $a : A$ is called "contractible":

**Definition**

We call $A$ **contractible** if we can construct a term of type

$$\mathsf{isContr}(A) := \sum_{(a:A)} \prod_{(x:A)} \mathsf{Id}(x, a)$$

# Homotopy levels

## Homotopy levels: the complete picture

$$\mathsf{isContr}(A) := \sum_{(a:A)} \prod_{(x:A)} \mathsf{Id}(x, a)$$

$$\mathsf{isProp}(A) := \prod_{x,y:A} \mathsf{isContr}(\mathsf{Id}(x, y))$$

$$\mathsf{isSet}(A) := \prod_{x,y:A} \mathsf{isProp}(\mathsf{Id}(x, y))$$

$$\vdots$$

$$\mathsf{isofhlevel}_{n+1}(A) := \prod_{x,y:A} \mathsf{isofhlevel}_n(\mathsf{Id}(x, y))$$

But we will not need the higher levels.

# Table of Contents

# Idea of Univalence : isomorphic types are equal

## Types are stratified in **universes**

- have a sequence of **universes** $(\mathcal{U}_n)_{n\in\mathbb{N}}$ (à la Russell)
- a universe $\mathcal{U}$ is a type
- any type $A$ is a point of some universe $A : \mathcal{U}$
- What does $\mathsf{Id}_{\mathcal{U}}(A, B)$ look like?

## **Univalence:** $\mathsf{Id}_{\mathcal{U}}(A, B) = (A \cong B)$

- Idea: any path $p : \mathsf{Id}(A, B)$ corresponds to an isomorphism $\bar{p} : A \xrightarrow{\sim} B$
- impose this correspondance as an axiom

# Isomorphism of types

## Definition (Isomorphism of types)

A function $f : A \to B$ is an **isomorphism of types** if there are

- $$g : B \to A$$
- $$\eta : \prod_{a:A} \mathsf{Id}\Big(g\big(f(a)\big), a\Big) \qquad \epsilon : \prod_{b:B} \mathsf{Id}\Big(f\big(g(b)\big), b\Big)$$

together with a coherence condition $\tau : \prod_{x:A} \mathsf{Id}\Big(f(\eta x), \epsilon(fx)\Big)$

# Isomorphism of types

**Definition (Isomorphism of types)**

A function $f : A \to B$ is an **isomorphism of types** if there are

- $$g : B \to A$$
- $$\eta : \prod_{a:A} \mathsf{Id}\Big(g(f(a)), a\Big) \qquad \epsilon : \prod_{b:B} \mathsf{Id}\Big(f(g(b)), b\Big)$$

together with a coherence condition $\tau : \prod_{x:A} \mathsf{Id}\Big(f(\eta x), \epsilon(fx)\Big)$

…ie. if we can construct a term of type

$$\mathsf{isIso}(f) := \sum_{(g:B\to A)} \sum_{(\eta:\_)} \sum_{(\epsilon:\_)} \prod_{(x:A)} \mathsf{Id}\Big(f(\eta x), \epsilon(fx)\Big)$$

# The type of isomorphisms

## Lemma

*For any* $f : A \to B$, *the type* $\mathsf{isIso}(f)$ *is a proposition. In particular, the inverse* $g$ *is* ***unique***, *if it exists.*

## Definition (Type of isomorphisms from $A$ to $B$)

$$\mathsf{Iso}(A, B) := \sum_{f : A \to B} \mathsf{isIso}(f)$$

- There are other, equivalent definitions of $\mathsf{isIso}(f)$.
- Isomorphisms of types are usually called “equivalences”.

# Examples of isomorphic types

**Example (Leibniz principle)**

For any $p : \mathsf{Id}(a, b)$, the substitution function

$$\mathsf{subst}_{a,b}(p) : C(a) \to C(b)$$

is an isomorphism with inverse $\mathsf{subst}_{b,a}(p^{-1})$.

- [True] is isomorphic to *Nat*
- propositions are isomorphic iff they are logically equivalent

# The elimination rule of the identity type

**The Identity elimination rule says:**

To define a function of type

$$\prod_{(x,y:A)} \prod_{(p:\mathsf{Id}(x,y))} C(x,y,p)$$

it suffices to specify its image on $(x, x, \mathsf{refl}(x))$.

# The Univalence Axiom

### Definition (From paths to isomorphisms)

$$\mathsf{idtoiso} : \prod_{A,B:\mathcal{U}} \mathsf{Id}(A, B) \to \mathsf{Iso}(A, B)$$

$$(A, A, \mathsf{refl}(A)) \mapsto (\lambda x.x, \_)$$

### Univalence Axiom

$$\mathsf{univalence} : \prod_{A\ B:\mathcal{U}} \mathsf{isIso}(\mathsf{idtoiso}_{A,B})$$

In particular, Univalence gives a map backwards:

$$\mathsf{isotoid}_{A,B} : \mathsf{Iso}(A, B) \to \mathsf{Id}(A, B)$$

# Consequences of Univalence

- Propositional extensionality

$$(P \leftrightarrow Q) \to \mathsf{Id}(P, Q)$$

- Function extensionality:

$$\prod_{x:A} \mathsf{Id}_B(fx, gx) \to \mathsf{Id}_{A \to B}(f, g)$$

  and its dependent variant

- Quotient types exist (cf. later)

# Table of Contents

# Categories in Univalent Foundations — Take I

## A naïve definition of categories

A **category** $\mathcal{C}$ is given by

- a type $\mathcal{C}_0$ of **objects**
- for any $a, b : \mathcal{C}_0$, a type $\mathcal{C}(a, b)$ of **morphisms**
- operations: identity & composition

$$\mathsf{id} : \prod_{a:\mathcal{C}_0} \mathcal{C}(a, a) \qquad (\circ) : \prod_{a,b,c:\mathcal{C}_0} \mathcal{C}(b, c) \times \mathcal{C}(a, b) \to \mathcal{C}(a, c)$$

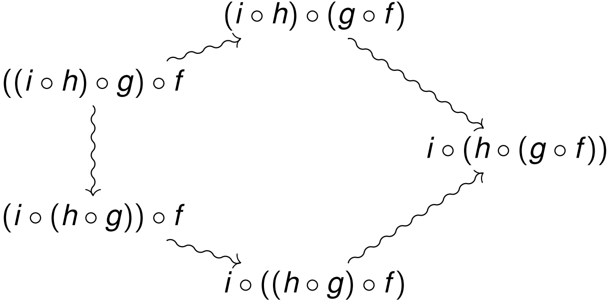
- axioms: unitality & associativity for any suitable $f, g, h$:

$$\text{unital} : \prod_{a,b:\mathcal{C}_0, f:\mathcal{C}(a,b)} (\mathsf{id}_b \circ f \rightsquigarrow f) \times (f \circ \mathsf{id}_a \rightsquigarrow f)$$

$$\mathsf{assoc} : \prod_{a,b,c,d,f,g,h} (h \circ g) \circ f \rightsquigarrow h \circ (g \circ f)$$

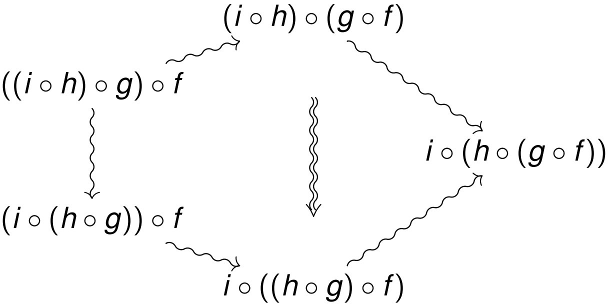
# Coherence for associativity – Mac Lane's pentagon

Problem with above definition: two ways to associate a composition of **four** morphisms from left to right:



# Coherence for associativity – Mac Lane's pentagon

Problem with above definition: two ways to associate a composition of **four** morphisms from left to right:




Would need to ask for higher coherence $\Rrightarrow$ , $\Rrightarrow$ etc

# Categories in Univalent Foundations — Take II

**Definition (Category in UF)**

A **category** $\mathcal{C}$ is given by

- a type $\mathcal{C}_0$ of objects
- for any $a, b : \mathcal{C}_0$, a $\boxed{\textbf{set}}$ $\mathcal{C}(a, b)$ of morphisms
- operations: identity & composition
- axioms: unitality & associativity

For this definition of category, all the postulated paths are trivially coherent.

# Isomorphism in a category

## Definition (Isomorphism in a category)

A morphism $f : \mathcal{C}(a, b)$ is an **isomorphism** if there are

- $$g : \mathcal{C}(b, a)$$
- $$\eta : g \circ f \rightsquigarrow \mathrm{id}_a \qquad \epsilon : f \circ g \rightsquigarrow \mathrm{id}_b$$

Put differently, we define

$$\mathrm{isIso}(f) := \sum_{g:\mathcal{C}(b,a)} \Big( (g \circ f \rightsquigarrow \mathrm{id}_a) \times (f \circ g \rightsquigarrow \mathrm{id}_b) \Big)$$

# Isomorphism in a category II

**Lemma**

*For any* $f : \mathcal{C}(a, b)$*, the type* $\mathsf{isIso}(f)$ *is a proposition.*

**Definition (The type of isomorphisms)**

$$\mathsf{Iso}(a, b) :\equiv \sum_{f:\mathcal{C}(a,b)} \mathsf{isIso}(f)$$

# What about categories as objects?

## Definition (Functor)

A **functor** $F$ from $\mathcal{C}$ to $\mathcal{D}$ is given by

- a map $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$
- for any $a, a' : \mathcal{C}_0$, a map $F_{a,a'} : \mathcal{C}(a, a') \to \mathcal{D}(Fa, Fa')$
- preserving identity and composition

## The category of categories?

- the type of functors from $\mathcal{C}$ to $\mathcal{D}$ does **not** form a set
- thus there is no category of categories

# Isomorphisms of categories

**Definition (Isomorphism of categories)**

A functor $F$ is an **isomorphism of categories** if

- $F_0$ is an isomorphism of types and
- $F_{a,a'}$ is an isomorphism of types (a bijection) for any $a, a'$,

$$\mathsf{isIsoOfCats}(F) :\equiv \Big(\ldots\Big) \times \Big(\prod_{a,a':\mathcal{C}_0} \ldots\Big)$$

# Isomorphism of categories II

## Lemma

*"Being an isomorphism of categories" is a proposition.*

## Definition (Type of isomorphisms of categories)

$$\mathcal{C} \cong \mathcal{D} := \sum_{F:\mathcal{C}\to\mathcal{D}} \mathsf{isIsoOfCats}(F)$$

# Natural transformations

**Definition (Natural transformation)**

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** $\alpha : F \to G$ is given by

- for any $a : \mathcal{C}_0$ a morphism $\alpha_a : \mathcal{D}(Fa, Ga)$ s.t.
- for any $f : \mathcal{C}(a, b)$, $Gf \circ \alpha_a \rightsquigarrow \alpha_b \circ Ff$

The type of natural transformations $F \to G$ is a **set**.

**Definition (Functor category $\mathcal{D}^{\mathcal{C}}$)**

- objects: functors from $\mathcal{C}$ to $\mathcal{D}$
- morphisms from $F$ to $G$: natural transformations

# Equivalence of categories

## Definition (Left Adjoint)

A functor $F : \mathcal{C} \to \mathcal{D}$ is a **left adjoint** if there are

- $G : \mathcal{D} \to \mathcal{C}$
- $\eta : 1_{\mathcal{C}} \to GF$
- $\epsilon : FG \to 1_{\mathcal{D}}$
- + higher coherence data.

# Equivalence of categories

### Definition (Equivalence of categories)

A left adjoint $F$ is an **equivalence of categories** if $\eta$ and $\epsilon$ are isomorphisms.

### Lemma

*"$F$ is an equivalence" is a proposition.*

### Definition

$$\mathcal{C} \simeq \mathcal{D} := \sum_{F : \mathcal{C} \to \mathcal{D}} \mathsf{isEquivOfCats}(F)$$

# Table of Contents

# From paths to isomorphisms

**Definition (From paths to isomorphisms, univalent categories)**

For objects $a, b : \mathcal{C}_0$ we define

$$\begin{aligned} \mathsf{idtoiso}_{a,b} : (a \rightsquigarrow b) &\to \mathsf{Iso}(a, b) \\ \mathsf{refl}(a) &\mapsto \mathsf{id}_a \end{aligned}$$

We call the category $\mathcal{C}$ **univalent** if, for any objects $a, b : \mathcal{C}_0$,

$$\mathsf{idtoiso}_{a,b} : (a \rightsquigarrow b) \to \mathsf{Iso}(a, b)$$

is an isomorphism of types.

# About univalent categories

- In a univalent category, isomorphic objects are equal.
- *"$\mathcal{C}$ is univalent"* is a proposition, written $\mathsf{isUniv}(\mathcal{C})$.
- Definition proposed by Hofmann & Streicher '98, but not pursued

# Examples of univalent categories

- <u>Set</u> (follows from the Univalence Axiom)
- categories of algebraic structures (groups, rings,...)
  - made precise by the **Structure Identity Principle** (P. Aczel)
- full subcategories of univalent categories
- functor category $\mathcal{D}^{\mathcal{C}}$, if $\mathcal{D}$ is univalent

# Some more examples of univalent categories

- a **preorder**, considered as a category, is **univalent** iff it is **antisymmetric**
- if $X$ is of h-level $3$, then there is a univalent category with $X$ as objects and $\mathsf{hom}(x, y) := (x \rightsquigarrow y)$
- if $\mathcal{C}$ is univalent, then the category of **cones** of shape $F : \mathcal{J} \to \mathcal{C}$ is
  - $\rightsquigarrow$ limits (limiting cones) in a univalent category are **unique**

# Non-univalent categories

- 

- more generally, any **chaotic** category $\mathcal{C}$ with $\mathcal{C}(x, y) := 1$ unless $\mathcal{C}_0$ is contractible

- any chaotic category $\mathcal{C}$ with an object $c : \mathcal{C}_0$ is **equivalent** to the terminal category $\mathbf{1} := \bullet$
  - ⇝ a category can be equivalent to a univalent one without being univalent itself

# 1 kind of sameness for univalent categories

| | |
|---|---|
| **Equality** | $\mathcal{C} \rightsquigarrow \mathcal{D}$ |
| **Isomorphism** | $\mathcal{C} \cong \mathcal{D}$ |
| **Equivalence** | $\mathcal{C} \simeq \mathcal{D}$ |

**Theorem**

*For **univalent** categories $\mathcal{C}$ and $\mathcal{D}$, these are isomorphic as types.*

**Consequence**

**Every property** of univalent categories definable in UF is **invariant under equivalence**.

# Table of Contents

# Rezk completion

- *"Being univalent"* is a proposition

$\rightsquigarrow$ Inclusion from univalent categories to categories

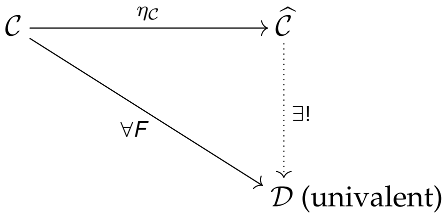
**Theorem**

*The inclusion of univalent categories into categories has a left adjoint (in bicategorical sense),*

$$\mathcal{C} \mapsto \widehat{\mathcal{C}}, \qquad \text{the } \textbf{Rezk completion} \text{ of } \mathcal{C} \quad .$$

# Rezk completion II

Any functor $F : \mathcal{C} \to \mathcal{D}$ with $\mathcal{D}$ univalent factors uniquely:




The functor $\eta_{\mathcal{C}}$ is the unit of the adjunction; it is

- fully faithful and
- essentially surjective.

# Construction of the Rezk completion

- $\widehat{\mathcal{C}}$ := full image subcat. of $\underline{\mathsf{Set}}^{\mathcal{C}^{\mathrm{op}}}$ of **Yoneda embedding**
  - $\widehat{\mathcal{C}}$ is univalent
- let $\eta_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ be the **Yoneda embedding** (into $\widehat{\mathcal{C}}$):
  - fully faithful
  - essentially surjective (by definition)
- precomposition $\_ \circ H : \mathcal{C}^{\mathcal{B}} \to \mathcal{C}^{\mathcal{A}}$ is an equivalence—and hence an isomorphism—of categories if
  - $H$ is essentially surjective
  - $\mathcal{C}$ is univalent
- the object function thus is an isomorphism of types

$$\_ \circ H : (\mathcal{C}^{\mathcal{B}})_0 \to (\mathcal{C}^{\mathcal{A}})_0$$

# Semantics of univalent categories

In Voevodsky's sSET model,

- categories correspond to truncated Segal spaces
- **univalent** categories correspond to truncated **complete** Segal spaces

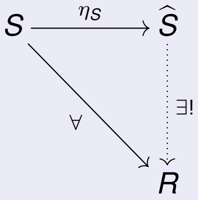
Completion for Segal spaces was studied by Rezk:

# Special case of Rezk completion: Quotienting

Specialise: category $\rightsquigarrow$ groupoid $\rightsquigarrow$ equivalence relation

**Theorem (Univalent Foundations admits quotients)**

*Any map* $f : S \to R$ *such that* $s \sim s' \implies f(s) \rightsquigarrow f(s')$ *factors uniquely via* $\widehat{S}$*:*




- More direct construction of set-level quotients by Voevodsky: "type of equivalence classes"

# Another example: the classifying space of a group

- Consider group $G$ as category with one element
- $\mathcal{B}(G)$ := classifying space, ie. the space such that

$$\Omega(\mathcal{B}(G)) = G$$

- Construction of $\mathcal{B}(G)$ as space of **torsors** is actually the process of Rezk completion
- Directly formalized in UF by Dan Grayson

# Mechanization in Coq

## Rezk Completion mechanized in Coq+UA+TypeInType

- approx. 4000 lines of code
- based on Voevodsky's library *"Foundations"*

## Design choices for the implementation

- Goal: make maths in UF accessible for mathematicians
  - $\rightsquigarrow$ stick to that part of syntax with clear semantics
- Restriction to basic type constructors ($\prod$, $\sum$,...)
- Coercions and notations as in mathematical practice
- No automation: no type classes, no automatic tactics

# Future work

## Towards higher categories

- no internal definition of $\infty$-categories
- 2 possible paths to higher categories:
  - "manual" definition of $n$-categories for low $n$
  - bootstrapping via enrichment in $n$-categories

## Requires notion/theory of

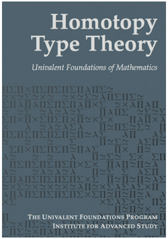
- enriched category theory and univalence
- truncation of higher categories

# Future work II

- Makkai: FOLDS (First Order Logic with Dependent Sorts) as foundation for category theory
- Goal: only invariant properties definable (no equality on objects)
- FOLDS embeds in type theory
- Suggested by Shulman: compare definition of univalent categories in FOLDS style to the one above

# References


- Univalent Foundations program, *Homotopy Type Theory: Univalent Foundations of Mathematics*, 2013



- Hofmann, M. and Streicher, T., *The groupid interpretation of type theory*, 1996
- Rezk, C., *A model for the homotopy theory of homotopy theory*, 2001
- preprint `arXiv:1303.0584`

Some background …

# A model of MLTT in simplicial sets

Types-as-spaces intuition is made precise by a model of MLTT:

- The category sSET of simplicial sets is Quillen-equivalent to the category TOP of topological spaces.
- There is a model of MLTT in simplicial sets (Voevodsky).
- This model satisfies an additional property: **univalence**
- This suggests adding univalence as an additional axiom (UA) to MLTT.

**Remark**

Traditional set-theoretic models of MLTT do not satisfy univalence and thus are not models of MLTT + UA.

# The groupoid interpretation of MLTT

## Hofmann & Streicher: independence of UIP

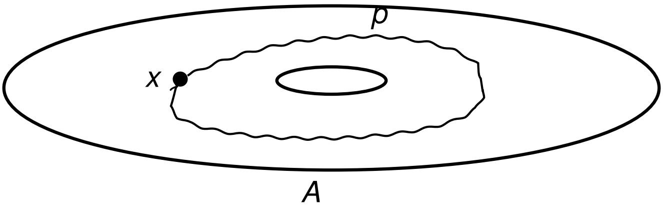
Given a type $A$, one can **not** construct a term of type

$$\prod_{(x:A)} \prod_{(p:\mathsf{Id}(x,x))} \mathsf{Id}_{\mathsf{Id}(x,x)}(p, \mathsf{refl}(x))$$

# Non-trivial loop spaces

**Interpretation of Hofmann & Streicher's result**

It is (equi-)consistent to have a type $A$ with non-trivial path spaces, e.g. a punctured disk.



# Truncation

## Propositional truncation

- to any type $A$ associate type $\|A\|_1$
- $\|A\|_1$ is a proposition
- $\|A\|_1$ indicates whether $A$ is inhabited or not, we have

$$A \to \|A\|_1 \qquad \neg A \to \neg \|A\|_1$$

- $\exists\, n : Nat, \mathrm{even}(n) := \|\sum_{n:Nat} \mathrm{even}(n)\|_1$

# Truncation

## Propositional truncation

- to any type $A$ associate type $\|A\|_1$
- $\|A\|_1$ is a proposition
- $\|A\|_1$ indicates whether $A$ is inhabited or not, we have

$$A \to \|A\|_1 \qquad \neg A \to \neg\|A\|_1$$

- $\exists\, n : Nat, \text{even}(n) := \|\sum_{n:Nat} \text{even}(n)\|_1$

## Truncation to homotopy level $n$

- similar truncation can be defined for any $n$, $A \mapsto \|A\|_n$
- $\|A\|_n$ has only trivial paths above level $n$

# Equivalent definitions of isomorphism

**Logically equivalent** definition:

- One possible $\tau$ can be deduced from $g$, $\eta$ and $\epsilon$

$\rightsquigarrow$ suffices to give $g$, $\eta$ and $\epsilon$ to prove that $f$ is an isomorphism

**But** the type of triples $(g, \eta, \epsilon)$ is **not** a proposition

Several equivalent definitions of isomorphism:

- "having a left- and a right-handed inverse"
- "having contractible fibers", i.e. inverse image of each point is a singleton